# Income Disaster Model with Optimal Consumption*

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**Abstract**

We propose a continuous-time income disaster model with optimal consumption. We endogenously determine the stochastic discount factor (SDF) in an incomplete market caused by income disaster. We then derive optimal consumption decisions for two types of agents, one who is exposed to income disaster and another who is not. We find a large incomplete-markets precautionary savings term between the two agents, which pushes the interest rate down and helps to resolve the risk-free rate puzzle. Interestingly, with income disaster the equilibrium interest rate is a decreasing function of risk aversion while the equity premium is an increasing function. Finally, our model can better match empirical marginal propensities to consume numbers and explain the low-consumption-high-savings puzzle.



Keywords: Income Disaster, Precautionary Savings, Incomplete Markets, Asset Pricing, Marginal Propensities to Consume

JEL Codes: D15, D58, G11, G12

# 1 Introduction

Half of jobs in the world today are susceptible to becoming automated in the future (Frey and Osborne, 2017). Potentially catastrophic loss of income is ubiquitous as differently evidenced by extreme events (e.g., the 2008 global financial crisis and the recent COVID-19 pandemic).[1] In particular,

> Around one third of jobs in the G20 countries rely directly on the effective management and sustainability of a healthy environment. Climate change and other forms of environmental degradation have already caused net negative impacts on jobs and work productivity, and these impacts are expected to become more pronounced in the coming decades. (International Labour Organization report, August 2018)

Climate change has driven an increase in concern about forced unemployment caused by global warming or extreme weather.[2] The above quote from the report by the International Labour Organization (ILO) demonstrates a matter of genuine urgency behind the study of income disaster. Along this line, this paper investigates how people could be encouraged to optimally adjust their consumption patterns especially with income disaster.

Considering such catastrophic income shocks triggered by various reasons as stated

[1]A key insight comes from economic recessions followed by human capital depreciation during long-term periods of unemployment. During the 2007-2009 Great Recession in the United States, many people experienced the unprecedented largest reductions in their consumption and unemployment. In the U.S., 44% of households were found to be unable to pay for an emergency expense of just $400 (Federal Reserve report, 2017), in the European Union, approximately 218 million people are experiencing earnings insecurity and volatility, thus struggling to ensure that future consumption needs can be met (European Commission statistics, 2017).

[2]According to the International Labour Organization report entitled "The employment impact of climate change adaptation," 23 million working-life years were lost annually at the global level between 2000 and 2015 as a result of various environment-related hazards (e.g., rainstorms, floods, forest fires and extreme weather). Global warming results in a decrease in labor productivity of jobs in farming, fishing and forestry, and all those depending upon natural processes.

above, we propose a continuous-time income disaster model with optimal consumption.[3] The model considers a continuous-time endowment economy with a one-time-only large negative Poisson shock to aggregate output.[4] The aggregate output is comprised of aggregate labor earnings and dividends. There are two types of agents who exhibit the constant relative risk aversion (CRRA) utility preference, one who receives income subject to the Poisson disaster shock and another who receives income unaffected by the disaster shock. The assets available for trade in the financial market include a risk-free asset and multiple risky assets. We endogenously determine the stochastic discount factor (SDF) in an incomplete market caused by income disaster.[5] We then derive optimal consumption decisions for the two agents.

We offer three insights helping the asset pricing and consumption/savings literature. We find a large incomplete-markets precautionary savings term between the two agents, which pushes the interest rate down and helps to resolve the risk-free rate puzzle. As far as income risks are concerned, such a precautionary savings motive is obtained only by income disasters with discrete and jump income shocks. The distinct feature of income disasters from regular income risks results from that the timing of disastrous income shocks is not deterministic, so its probability changes in time.[6] Concretely, income disaster occurs

[3]We have established in Section A in Appendix a simple two-period model with a jump-type income shock and obtained closed-form solutions with the simple quadratic utility function. However, solving analytically or even numerically the model in multi periods with the constant relative risk aversion utility function turns out to be a considerable challenge.

[4]We relax this restrictive modeling assumption in our extension exercise in Section 6 by allowing for multiple Poisson shocks with time-varying state-dependent intensity.

[5]It has long been known that market completeness under no arbitrage implies the existence of a unique SDF and the resulting unique risk-neutral measure. However, when markets are incomplete, the number of SDFs is infinite, so the set of equivalent martingale measures is also infinite. To price the expected return on an asset in an incomplete market, this multitude of SDFs must be pruned to one. We find a way to construct the unique risk-neutral measure with the uniquely determined risk-neutral income disaster intensity.

[6]We allow in Section 6 for the timing of income disasters evolves in time even stochastically, leading to time-varying state-dependent income disasters.

at an uncertain time, which is assumed to be the first jump time of an independent Poisson process. This random time of income disaster introduces a new friction into the economy and has implications for asset pricing.

We find that income risks result in a nonstandard form of market incompleteness especially when those involve an uncertain time-horizon like income disasters considered in this paper. Income risks modeled by uninsurable Brownian motions give rise to additional precautionary savings due to the randomness of labor income levels (labor market risk). The labor market risk induces in the SDF a new market price component for uninsurable income risks. However, the new market price of uninsurable income risks operate with income risks-related Brownian motions only, so it does not involve any adjustment of the risk-free interest rate in the SDF's drift term.[7] As a result, a regular Brownian-type diffusive and continuous income shock could not generate the substantial precautionary savings that matter for general equilibrium quantities. In contrast, income disasters modeled by Poisson shocks give rise to a sufficiently strong demand for optimal savings that reduces the risk-free rate significantly in equilibrium. The specific timing risk exists with income disasters in addition to the randomness of labor income levels and it induces in the SDF a specific form of the market price for income disasters, which alters the drift of the SDF with adjustments of the risk-free interest rate.[8]

Second, with income disaster the equilibrium interest rate is a decreasing function of risk aversion while the equity premium is an increasing function, helping to disentangle the risk-free rate puzzle from the equity premium puzzle. In the standard CRRA representative asset pricing models without income disaster, the equilibrium interest rate is typically an

[7]The equilibrium risk-free interest rate is therefore not altered by unspanned Brownian-type income risks. This is also known as the irrelevance result by Krueger and Lustig (2010). We confirm this irrelevance result in our income disaster model by allowing for extra undiversifiable Brownian income risk in Section 6.

[8]Here, the specific form of the market price refers to the nonstandard form of SDF as compared to SDFs with respect to market risks. Theorem 3.1 in Section 3 explicitly characterizes such a nonstandard form of the SDF with respect to income disaster.

increasing function of risk aversion[9] due to the effects of the elasticity of intertemporal substitution (EIS), as pointed out by Weil (1989).[10] Thus, the existing CRRA models pose serious challenges to match both low risk-free rate and high equity premium simultaneously, because the equity premium is generally an increasing function of risk aversion. In fact, the equity premium is still an increasing function of risk aversion in the presence of income disaster. However, the precautionary savings motive is so strong that for typical parameters the equilibrium interest rate is a decreasing function of risk aversion, dominating the effects of the EIS.

Third, the model can better match empirical marginal propensities to consume (MPC) numbers and explain the low-consumption-high-savings puzzle. The model's ability with income disaster to match the MPCs is much better than without income disaster, generating with just 1% income disaster possibility as high as 8% in some reasonable scenarios that is two times higher than 4% MPC that most macroeconomic models theoretically suggest.[11]

Intuitively, the substantial precautionary savings required by our model with income disaster allow for large consumption responses to changes in wealth, as opposed to the low sensitivity of consumption to wealth changes suggested by classical consumption/savings models. Our equilibrium consumption growth drift is an increasing function of income

[9]More precisely, the risk-free rate in the absence of income disaster is a quadratic function of relative risk aversion and volatility of consumption growth rate. However, given the square of consumption volatility approaches nearly zero (e.g., 3% consumption volatility approaches 0.09%(=0.00009)), the risk-free rate turns out to rise in a linear way with respect to an increase in risk aversion, thus generating its increasing monotonicity with risk aversion.

[10]As far as the CRRA utility preference is concerned, the EIS is the inverse of the coefficient of relative risk aversion, so that a higher risk aversion results in a lower EIS. The lower EIS implies that the agent's optimal decision is to consume more immediately and save less, because consumption would not grow higher than as expected with respect to an increase in the interest rate.

[11]While most macroeconomic models theoretically suggest around 4% MPCs, the empirically observed MPCs range from 20% to 60% (Carroll *et al.*, 2017) and about 10% MPCs can be regarded as the low end of the MPC range (Fisher *et al.*, 2019). According to Parker *et al.* (2013), the empirical MPC estimates range from 12% to 30%.

disaster possibility and thus, the so-called excess sensitivity puzzle in consumption can be safely ignored. More specifically, higher chances of being caught up with income disaster the agent is exposed to, she has to pay more for the same amount of future consumption at the expense of relatively more expensive present consumption, thus requiring the agent's greater consumption responses, MPCs, to changes in wealth.

In line with empirical and anecdotal evidence for disastrous income risk, we focus on the extremes of the probability distribution of income by income disaster, deviating from log-normality substantially.[12] In terms of the modeling contribution, incorporating jumps (even one event Poisson jump) into labor income leads to incomplete markets, and it is in general very difficult to get analytical solutions to problems in incomplete markets.[13] We significantly extend existing (martingale) methods by Cox and Huang (1989) and Karatzas *et al.* (1991) (who do not allow for income and its jump risk) to solve incomplete market partial equilibrium consumption/savings and portfolio choice problems. The extension that we consider in the paper uses ideas by Bensoussan *et al.* (2016) to endogenously determine the unique SDF in incomplete markets.[14] The model yields analytical solutions for the SDF and consequently, the optimal consumption choice as a function of the SDF. We then obtain the analytically tractable equilibrium interest rate and equity premium.

One very simple message delivered by our income disaster model is that the agent who encounters income disaster consumes less and saves more than the agent who does not. As an extreme simplification to capture consumption/savings behaviors of households in

[12]Standard literature relying on Brownian risks with log-normality fails to take sufficient account of the low-probability, high-impact aspect of income disaster. However, large and negative earnings losses are observed at job displacement (Low *et al.*, 2010), and such substantial income losses should have a large impact on investment and consumption/savings choices (Guvenen *et al.*, 2015).

[13]One may encounter partial integro-differential equations in incomplete markets caused by income disaster, which are difficult to solve analytically in general.

[14]Our incomplete-market optimization problem could be solved only if the unique SDF in the incomplete market was determined in order to find the so-called minimal local martingale measure (Karatzas *et al.*, 1991). Here, such an unique determination could be attained by using the dynamic programming approach of Bensoussan *et al.* (2016) especially with its analytically tractable solution to the non-linear differential equation derived in the incomplete market.

developing countries, we assume that the agent with high disaster exposure represents developing countries that rely heavily on the production of commodities whose prices are more prone to disaster-like movements. Relatively, the agent with low disaster exposure is assumed to represent developed countries that handle most of the manufacturing. In the context of merely an illustrative application, the simple message with income disaster is helpful for understanding the low-consumption-high-savings rates of households in developing countries. Households in developing countries find income disaster much harder to buffer than those in developed countries, unless there is sufficient insurance against income disaster. So, readiness for income disaster requires a large amount of precautionary savings for households in developing countries, cutting back their consumption in the event of a sudden fall in income caused by income disaster.

The research is contributing to two very mature literatures: (i) incomplete-market consumption/savings literature on the macro side and (ii) equilibrium representative-agent asset pricing literature on the finance side. More specifically, we share ingredients with the partial equilibrium literature with incomplete markets in the sense that cash flows on financial assets are assumed to be orthogonal to income disaster, making markets incomplete. We also share ingredients in common with the general equilibrium literature with rare disaster in the sense that the shock is common, hitting all exposed agents at the same time.

Given the markets are typically incomplete with uninsurable labor income shocks, the classical martingale approach (Cox and Huang, 1989) that uses the risk-neutral measure is no longer available in the baseline investment and consumption/savings models. In order to address the challenges of market incompleteness, instead of the martingale pricing approach, the alternative dynamic programming approach can be used in incomplete markets (Duffie *et al.*, 1997; Liu *et al.*, 2005). However, in this case, it involves highly non-linear Hamilton-Jacobi-Bellman (HJB) equations, which are very difficult to solve analytically. Therefore, the use of dynamic programming approach requires complex numerical schemes to solve incomplete market problems. With no consideration of labor income and its risk, one can adopt the approaches of Garlappi and Skoulakis (2010), Jin and Zhang (2012), and Jin *et al.* (2017) for such a numerical approach to solving consumption and investment problems in incomplete markets.

Bensoussan *et al.* (2016) first study a partial equilibrium model of optimal consumption/savings, investment, and retirement with jump-type forced unemployment risk. They study the logarithmic utility without borrowing constraints in the partial equilibrium. Here, we significantly extend the idea in Bensoussan *et al.* (2016) with the (more general) CRRA utility and the borrowing constraints in the general equilibrium. Our paper then incorporates income disaster in a general equilibrium setting with both the returns of risky assets and the risk-free rate endogenously determined; and, more importantly, we study the model implications in terms of the risk-free rate puzzle and the MPCs.

The precautionary savings literature is vast in macro. With an emphasis on the MPC, the standard consumption/savings models (e.g., Bewley, 1977; Campbell, 1987; Caballero, 1990; Wang, 2003) imply MPCs typically around 4%. While empirically plausible MPC estimates range from 10% to even 60%.[15] Contrary to insignificant role of income risk in matching empirical MPCs by the standard models,[16] Wang *et al.* (2016) study an incomplete-market consumption-savings model with recursive utility and stochastic income modeled by both a Brownian motion and large jumps in labor income; their paper focuses on the partial equilibrium with the interest rate being fixed, even without risky assets. Wang *et al.* (2016) improve their model's matching ability in empirical MPC numbers significantly by joint consideration of stochastic income with its jumps and borrowing constraints. We also isolate and very closely investigate the effects of such an income jump shock with borrowing constraints on the MPC, especially in the general equilibrium (not in the partial equilibrium as most studies do). The substantial optimal savings implied by Wang *et al.* (2016) or our model carry over to the general equilibrium and thus, equilibrium

[15]Parker *et al.* (2013): 12% to 30%, Carroll *et al.* (2017): 20% to 60%, Fisher *et al.* (2019): 10%.

[16]According to the consumption/savings predictions of Bewley (1977) and Campbell (1987), an income shock is less likely to affect the optimal savings of people. This is because consumption of people can be financed mainly by total available financial resources (i.e., total wealth consisting of financial wealth and human capital) rather than financial wealth only. Intuitively, the ability to self-insure against the income shock improves as long as total wealth is large.

MPCs with income disaster can be much higher than those implied by the standard models without income disaster, attempting to match the empirical MPC using the model.

On the other hand, it has long been known that households in developing countries are inclined to consume less and save more than those in developed countries (Cao and Modigliani, 2004). A large gap in savings rates between developing and developed countries gives rise to the so-called the low-consumption-high-savings rate puzzle. For instance, the household savings rate in China is on average 20% of disposable income from 1989 to 2009, while it is merely 4% in the United States over the same period (Choi *et al.*, 2017). Possible explanations to help understand this puzzle include, but not limited to the role of insurance in household savings decisions (Elmendorf and Kimball, 2000; Gormley *et al.*, 2012), the relation between demographic patterns and household savings rates (Cao and Modigliani, 2004; Curtis *et al.*, 2015), precautionary savings motives (Choi *et al.*, 2017; He *et al.*, 2018), housing wealth (Chen *et al.*, 2017; Painter *et al.*, 2022). In this paper, we support the result of Choi *et al.* (2017) and He *et al.* (2018) by taking a different route that income disaster could play a role in resolving the puzzle by highlighting how exposure to income disaster affects household savings decisions with precautionary savings motives in both partial equilibrium and general equilibrium.

On the asset pricing side, we are building a general equilibrium model with two heterogeneous agents. The general equilibrium models with heterogeneous agents have been established to help understand empirical patterns in the data.[17] There are related researches on asset pricing with unhedgeable income risk without income disaster (or jumps): Lucas (1994), Gomes and Michaelides (2008), and Constantinides and Ghosh (2017). Generalizing Constantinides and Duffie (1996), Schmidt (2016) thoroughly investigates asset pricing implications of tail risk in consumption growth and income with recursive preferences,

[17]For instance, these models have been used to understand the empirical properties of: excessive stock market volatility (Dumas *et al.*, 2009), liquidity and asset prices (Osambela, 2015), aggregate investment, consumption, output and equity prices (Baker *et al.*, 2016), international finance anomalies such as the co-movement of returns and capital flows or home-equity preference (Dumas *et al.*, 2017), and the yield curve (Ehling *et al.*, 2018).

income skewness, heterogeneous agents, and incomplete markets. None of these studies, however, endogenize adjustments in both stock market and consumption as we do with income disaster.

We also draw on the literature on the risk-free rate puzzle. Weil (1989) has first identified that a very high interest rate is necessarily obtained in an attempt to match the empirical equity premium with a very high risk aversion because both the equity premium and the risk-free rate in standard asset pricing models are increasing functions of risk aversion. Lucas (1994) has then demonstrated the marginal contribution of unhedgeable income risks to match the risk-free rate. Consistent with Lucas (1994), Krueger and Lustig (2010) have obtained the similar result that uninsurable Brownian-type income shocks are irrelevant or the risk-free rate, which is the irrelevance result. Contrary to Lucas (1994) and Krueger and Lustig (2010), Christensen *et al.* (2012) have highlighted the incremental contribution of unhedgebale Brownian-style income risks to match the risk-free rate especially when agents have heterogeneous risk aversion preferences. Constantinides and Ghosh (2017) have addressed the risk-free rate puzzle by shocks to household consumption growth especially through the time-varying idiosyncratic labor income risk. Recently, AI and Bhandari (2021) have thoroughly investigated the role of uninsured tail risk in labor income for asset pricing in the optimal risk-sharing context and obtained a high equity premium and a low risk-free rate with a moderate level of risk aversion, addressing the equity premium and risk-free rate puzzles.[18] In line with the income risk channel, we contribute to the literature by studying the impact of income disaster on the risk-free rate especially with optimal consumption/savings and investment choices that are considered in the general equilibrium constraints.

This paper is, in particular, similar to the rare disaster literature. Income disaster

[18]There are different perspectives to resolve the risk-free rate puzzle. Bansal and Yaron (2004) have established the long-run risk model where long-run shocks to consumption are priced for asset prices including the equity premium and the risk-free rate. With an emphasis on parameter uncertainty, Weitzman (2007) and Collin-Dufresne *et al.* (2016) have shown the significant impact of parameter learning about associated probabilities to explain the empirical level of risk-free rate.

includes a low-probability, depression-like third state of Rietz (1988)'s model in the individual's income process, which can be regarded as a different application of the rare disaster risk hypothesis by Rietz (1988). The rare disaster hypothesis arguably states that the slim chance of rare disasters (e.g., economic crisis or war) can dominate the determinaiton of asset risk premia. The seminal work of Rietz (1988), Barro (2006) and Gabaix (2008, 2012), Wachter (2013), Farhi and Gabaix (2016), and others have established different versions of the rare disaster hypothesis, thereby explaining empirical regularities such as the equity premium puzzle and the risk-free rate puzzle. Focusing on a firm's investment decision in a complete market (not the individual's investment decision in the incomplete market), Pindyck and Wang (2013) develop a general equilibrium model to study how rare disasters affect a central planner's consumption decision and asset pricing. They demonstrate an additional negative effect of the presence of rare disasters on the MPCs and the equilibrium risk-free rate. Rare disasters may be generated by various reasons such as climate change, weather disasters, business cycles, labor market frictions, etc. In this context, Hong *et al.* (2023) take Bayesian learning into account disaster intensity to consider the realistic reasons of rare disasters. More recently, Barro *et al.* (2022) establish a model of rare disasters with heterogeneous risk aversion and recursive utility, and investigate the quantity of safe assets. Consistent with the rare disaster hypothesis, the possibility of income disaster considered in this paper can account for high risk premium on bonds especially through the precautionary savings channel implied by the agent's incomplete-market optimal consumption choice.

The reason why the risk-free rate is low differs from the rare disaster models. Similar to the disaster models, both labor earnings and dividends fall sharply upon a disaster realization. Different from the disaster models in which the SDF is exogenously given by the marginal utility of consumption, the SDF in our model is endogenously determined with the disaster-exposed agent's optimal consumption choice in an incomplete market. Hence, the agent's incomplete-markets precautionary savings motive through her optimal consumption decision affects the SDF and leads the equilibrium risk-free rate to decrease accordingly.

This paper is organized as follows. After the basic model setting in Section 2, we provide in Section 3 analytic results for the endogenously determined SDF and the optimal consumption choice. We then establish the income disaster model with optimal consumption in Section 4. Quantitative analysis is given in Section 5. We discuss in Section 6 general cases with multiple Poisson shocks driven by time-varying state-dependent intensity and with uninsurable diffusive and continuous income shocks. In Section 7, we conclude the paper.

## 2 Basic Setting

We first lay out the mathematical building blocks for the uncertainty structure. The Brownian-uncertainty in the economy is modeled by the complete probability space $(\Omega, \mathcal{A}, P)$ on which the multi-dimensional Brownian motion process $Z(t)$ used in the stock prices (2) is defined. The probability space $(\Omega, \mathcal{A}, P)$ is captured by the filtration $\mathcal{F} = \{\mathcal{F}_t; t \geq 0\}$ which is the usual $P$-augmentation of $\sigma(Z(s); 0 \leq s \leq t)$ generated by the standard Brownian motion process $Z$. All statements including random variables are understood to hold in the mathematical context of either almost everywhere or almost surely.

The aggregate output process $I(t)$ is modeled by a geometric Brownian motion with one Poisson shock as follows:

$$dI(t) = \mu^I I(t-)dt + (\sigma^I)^\top I(t-)dZ(t) - (1-k)I(t-)dN(t), \quad I(0) = I > 0, \qquad (1)$$

where $\mu^I$ is the output mean, $\sigma^I$ is the standard deviation vector, $Z(t)$ is the market factor considered in the stock prices (2), and $k \in (0,1)$ is the recovery parameter. Here, $N(t)$ is a one-time Poisson shock with intensity $\delta > 0$:

$$dN(t) = 1, \quad t \geq \tau; \qquad dN(t) = 0, \quad t < \tau,$$

where $\tau$ is an exponential random variable with intensity $\delta$.[19]

[19]Notice that the random arrival $\tau$ of income disaster is not assumed to be a stopping time of the filtration $\mathcal{F}$ generated by asset (stock or income) prices. Rather, the date $\tau$ is randomly taken to be a

There are two information sets available to the agent. First, at any time $t$ the agent knows information about past values of stock prices ($\mathcal{F}_t$). Second, the agent also knows information about whether income disaster has occurred or not ($\mathcal{N}_t \equiv \sigma(\tau \wedge t)$), where $\tau \wedge t$ represents $\inf(\tau, t)$, which is the smallest filtration satisfying the so-called progressive enlargement of $\mathcal{F}$ with respect to $\tau$, which is denoted by $\mathcal{G}$.

**Definition 2.1** *The progressive enlargement of $\mathcal{F}$ with respect to a random time $\tau$, denoted by $\mathcal{G} = \{\mathcal{G}_t; t \geq 0\}$, is the smallest filtration including $\mathcal{F}$, where $\tau$ is a stopping time. Introduce $\mathcal{N}_t = \sigma(\tau \wedge t)$, which is the filtration generated by the family $\tau \wedge t$. By definition, the filtration $\mathcal{G}$ is taken to be the smallest right continuous family of sigma-fields such that both $\mathcal{F}_t$ and $\mathcal{N}_t$ are in $\mathcal{G}_t$.*

Notice that if $\tau$ is assumed to be a $\mathcal{F}$-stopping time, this enlargement reduces to $\mathcal{G} = \mathcal{F}$.

There is one riskless bond and multiple risky stocks. The bond price $B$ and the stock prices $S$ are given by

$$dB(t) = rB(t)dt, \qquad dS(t) + d(t)dt = S(t)\{\mu dt + \sigma^\top dZ(t)\}, \tag{2}$$

where $r$ is the risk-free interest rate, $d(t) = (d_1, ..., d_N)^\top$ are dividends for $N$ risky stocks, $\mu$ is the mean vector, $\sigma$ is the nonsingular standard deviation matrix, and $Z(t)$ is the standard Brownian motion process with the dimension equal to the number of linearly independent returns on stocks. Notice that the risk-free interest rate $r$, the mean vector $\mu$, and the standard deviation matrix $\sigma$ are to be determined from equilibrium conditions (Section 4).

positive value according to its exponential distribution with intensity $\delta$, so it is measurable with respect to the sigma-algebra $\mathcal{A}$. Put differently, observing asset prices up to time $t$ does not imply full information about whether $\tau$ has occurred or not by time $t$. Hence, there are some dates $t \geq 0$ such that the event $\{t < \tau\}$ is not $\mathcal{F}_t$-measurable. In this paper, income disaster is assumed to be driven by a one-time Poisson shock with constant $\delta$, thus rather assuming the independence that income disaster occurrence (the first jump time of the Poisson process) is not related to stock prices at all, i.e., $P[\tau > t|\mathcal{F}_\infty] = P[\tau > t]$ for all $t$, where $\mathcal{F}_\infty$ includes all possible information about the whole path of stock prices.

Following Krueger and Lustig (2010), Garleanu and Panageas (2015), and Barro *et al.* (2022), the fraction $\xi \in (0,1)$ of aggregate output constitutes aggregate labor earnings $\xi I(t)$.[20] The remaining fraction $1-\xi$ of aggregate output is then paid out as a dividend as follows: $D(t) \equiv \sum_{i=1}^{N} d_i(t) = (1-\xi)I(t) = I(t) - \xi I(t)$, which shows that the presence of income disaster (in the form of the Poisson shock) would, thus, affect asset returns (2) as well.[21]

We consider an infinite-horizon economy with a single consumption good (the numeraire). Each representative agent has wealth $W(t)$ and invests $\pi(t)$ in the stock market, and saves her remaining wealth $W(t) - \pi(t)\mathbf{1}$ in the bond market, where $\mathbf{1}$ is a vector of one's with dimensionality equal to the number of stocks, $\pi(t)$ is the dollar amount vector invested in each risky stock.[22] The agent also consumes $c(t)$ and receives $\xi I(t)$. The agent's dynamic wealth (budget) constraint is then: $W(0) = w > -\xi I/\beta_1$,

$$dW(t) = \{rW(t) - c(t) + \xi I(t) + \pi(t)^\top(\mu - r\mathbf{1})\}dt + \pi(t)^\top \sigma^\top dZ(t), \tag{3}$$

The borrowing limit is imposed by the following wealth constraints:

$$W(t) > -L(t) \geq -\frac{\xi I(t)}{\beta_1}, \quad \text{for all } t \in [0,\tau], \qquad W(t) > -\frac{k\xi I(t)}{\beta_1}, \quad \text{for all } t > \tau, \tag{4}$$

where $L(t)$ ($L(0) = L > 0$) is a non-negative time-varying function exogenously given,

$$\beta_1 = r - \mu^I + (\sigma^I)^\top \theta, \; \theta = (\sigma^\top)^{-1}(\mu - r\mathbf{1}). \tag{5}$$

Without $k$, the wealth constraint (4) is same with the free borrowing against wages; in other words, the agent is allowed to borrow against the present value of her future wages

[20] According to Krueger and Lustig (2010), there exists a Lucas tree generating a share of aggregate output as capital income, i.e., dividends, so that the remaining share represents labor earnings. Garleanu and Panageas (2015) have adopted this kind of aggregate output economy, where the aggregate earnings are given by a fraction of aggregate output and the remaining fraction of output is being paid out as dividends. Barro *et al.* (2022) have also worked with this Lucas-tree world.

[21] Aggregate earnings risks and asset return risks are closely related (Ball *et al.*, 2009). For instance, there is a positive relation between earnings and asset returns at the firm level (Ball and Brown, 1968).

[22] Our focus until Section 3 is on the impact of income disaster on a single agent's optimal consumption and investment policies. We then consider a general equilibrium analysis in Section 4 for two representative agents, the normal agent who is free from income disaster and the income-disaster-exposed agent.

(or the human capital).[23] The recovery parameter $k$ is needed in (4) to accommodate the jump in $I(t)$. The similar type of constraint was proposed in Bensoussan *et al.* (2016).

Note that since we only consider one Poisson event, after the the arrival of the jump event the aggregate earnings are reduced to $k\xi I(t)$ from $\xi I(t)$, where the aggregate output $I(t)$ follows a geometric Brownian motion without jumps: $dI(t) = \mu^I I(t)dt + (\sigma^I)^\top I(t)dZ(t)$, $I(0) = I > 0$.

The agent's optimal consumption and investment model is to maximize over infinite horizon her CRRA utility from intermediate consumption with wealth constraints (3) and (4) by optimally controlling her consumption $c$ and investment $\pi$.[24] The value function is given by

$$V(w, I) \equiv \sup_{(c,\pi)} E\Big[\int_0^\tau e^{-\beta t}\frac{c(t)^{1-\gamma}}{1-\gamma}dt + e^{-\beta\tau}\int_\tau^\infty e^{-\beta(t-\tau)}\frac{c(t)^{1-\gamma}}{1-\gamma}dt\Big], \tag{6}$$

subject to the wealth constraints (3) and (4), where $\beta > 0$ is the agent's subjective discount rate which can incorporate a hazard rate of death by standard arguments and $\gamma > 0$ ($\gamma \neq 1$) is the agent's constant coefficient of relative risk aversion.

After the arrival of the Poisson shock, the value function reduces to that in Merton (1969, 1971) except with the initial value $I(0)$ being replaced by $kI(0)$, because the agent's optimal consumption and investment strategy follows Merton's strategy with the standard wealth constraint without jumps (i.e. with $\tau = 0$ in (4)) and the new initial value. More precisely,

$$V^A(w, kI) \equiv \sup_{(c,\pi)} E\Big[\int_0^\infty e^{-\beta t}\frac{c(t)^{1-\gamma}}{1-\gamma}dt\Big] = K\frac{(w + k\xi I/\beta_1)^{1-\gamma}}{1-\gamma}, \tag{7}$$

[23]Following Koo (1998), the certainty equivalent present value (CEPV) of lifetime labor income can be defined with the standard SDF with which a zero risk premium is obtained for any risk orthogonal to the stock market risks. The CEPV then serves as the human capital that is the present value of the agent's future wages. The detailed calculations for the human capital value based on the CEPV are available in Section B in Appendix.

[24]Throughout the paper, we only consider the set of admissible policies of consumption $c(t)$ and investment $\pi(t)$ satisfying the dynamic wealth constraint given in (3) and the wealth constraints given in (4).

with $K = A^{-\gamma}$, $A = \frac{\gamma - 1}{\gamma}\Big(r + \frac{||\theta||^2}{2\gamma}\Big) + \frac{\beta}{\gamma}$. Note that with the new initial condition $kI(0)$ the constraint (4) is satisfied after $\tau$, which is the standard wealth constraint without jumps.

By the principle of dynamic programming, the value function given in (6) can be rewritten as the following:

$$V(w, I) = \sup_{(c,\pi)} E\Big[\int_0^{\tau} e^{-\beta t}\frac{c(t)^{1-\gamma}}{1-\gamma}dt + e^{-\beta\tau}V^A(W(\tau), kI(\tau))\Big].$$

Integrating out the random arrival $\tau$ allows us to restate the optimal consumption and investment problem as the following:[25]

$$V(w, I) = \sup_{(c,\pi)} E\Big[\int_0^{\infty} e^{-(\beta+\delta)t}\Big(\frac{c(t)^{1-\gamma}}{1-\gamma} + \delta K\frac{\{W(t) + k\xi I(t)/\beta_1\}^{1-\gamma}}{1-\gamma}\Big)dt\Big], \tag{8}$$

subject to (3) and (4).

Note that the agent with income disaster becomes forward looking and aims to maximize her consumption utility before income disaster occurs, and incorporating the value function after income disaster occurs in the optimization with the income disaster intensity $\delta$.

# 3 Analytic Results

The challenge of solving the agent's problem in incomplete markets results from a source of indeterminacy of SDFs. To put it another way, the existence of a unique SDF is not straightforward in incomplete markets. To address the challenge, we endogenously determine the unique SDF in two steps. First, we explicitly characterize SDFs in incomplete markets. Next, we determine the unique SDF with which the unique risk-neutral measure with respect to income disaster can be constructed so that the expected return on the agent's wealth becomes the risk-free rate. All the technical details behind the derivation and some related notations are available in Section D and Section E in Appendix.

[25]Such a method when solving a standard stochastic control problem with random horizon has been utilized widely by Liu and Loewenstein (2002) and Lin et al. (2022) more recently. For the details of the derivation, refer to Section C in Appendix.

**Theorem 3.1** *The unique stochastic discount factor (SDF) in an incomplete market caused by income disaster is endogenously determined as*

$$\xi^{\hat{\delta}}(t) = \exp\left\{ \ln\left(\frac{\hat{\delta}}{\delta}\right) N(t) - (\hat{\delta} - \delta)t \right\} H(t), \tag{9}$$

*where* $\hat{\delta}$ *is the unique risk-neutral income disaster intensity given by*

$$\hat{\delta} = \left(\frac{w}{\xi I} + \frac{k}{\beta_1}\right)^{-\gamma} \frac{\delta K}{z},$$

$z$ *is the dual variable corresponding to financial wealth* $w$ *by their relation as follows*

$$w + \frac{\xi I}{\beta_1} = \frac{\xi I}{\hat{A} + \delta/\gamma} z^{-1/\gamma} + \xi I B^*_{\delta} z^{-\alpha^*_{\delta}} + IP, \tag{10}$$

$\hat{A}$ *is defined in (A-26),* $-1 < \alpha^*_{\delta} < 0$ *is one root of the characteristic equation in (A-28),* $B^*_{\delta}$ *is a constant to be determined by (A-27),* $IP = IP1 + IP2$ *represents the integral parts given by (A-35), and the dynamics of* $H(t)$ *are given by*

$$dH(t) = -H(t)\{rdt + \theta^{\top} dZ(t)\}, \quad H(0) = 1.$$

The endogenously determined SDF given in (9) is a generalized version of the well-known Arrow-Debreu price. The identified quantity $\xi^{\hat{\delta}}(t)$ can be regarded as the Arrow-Debreu price per unit probability of one unit consumption good in state at time $t$.[26] The derived SDF determines the unique risk-neutrality with respect to income disaster. It can be, thus, used as the Randon-Nikodym derivative for measure change purposes in pricing interesting claims that are exposed to income disaster.

Importantly, $\hat{\delta}$ given in the theorem is the so-called risk-neutral income disaster intensity and the market price of income disaster can be defined as the logarithm of the ratio of risk-neutral intensity $\hat{\delta}$ to original intensity $\delta$, i.e., $\ln(\hat{\delta}/\delta)$, in addition to the canonical market price of risk (or the Sharpe ratio) $\theta$. Using Itô's formula, we obtain before income disaster occurs the following dynamics of the SDF given in Theorem 3.1: for $t < \tau$,

$$\begin{aligned} d\xi^{\hat{\delta}}(t) &= -\xi^{\hat{\delta}}(t)[\{r + (e^{\ln(\hat{\delta}/\delta)} - 1)\delta\}dt + \theta^{\top} dZ(t)] \\ &= -\xi^{\hat{\delta}}(t)[\{r + (\hat{\delta} - \delta)\}dt + \theta^{\top} dZ(t)], \end{aligned} \tag{11}$$

[26] Basically, the Arrow-Debrew price is the equilibrium price of one unit consumption good. It can serve as a shadow price for discounting future costs and benefits in financial analysis.

which shows the impact of income disaster affecting the SDF with adjustment of the risk-free interest rate by the magnitude of market price of income disaster $\ln(\hat{\delta}/\delta)$, i.e., the drift of the SDF increases from the risk-free rate by the difference $\hat{\delta} - \delta$ between risk-neutral income disaster intensity $\hat{\delta}$ and original intensity $\delta$.

Theorem 3.1 allows a convenient multiplicative separation of the traditional Arrow-Debreu price and the income disaster adjustments. In the absence of income disaster ($\delta = \hat{\delta}$), the SDF (9) reduces to the conventional Arrow-Debrew price and presents only the aggregate output uncertainty adjustments without any variations in the risk-free interest rate as illustrated by (11). In the presence of income disaster with the positive market price of income disaster, i.e., when $\hat{\delta} > \delta$,[27] the increased drift by the market price of income disaster in the SDF dynamics (11) results in more expensive equilibrium consumption price in the future as identified in (9). Hence, the income-disaster-exposed agent is in a high marginal utility state and hence willing to give up more consumption than without income disaster in order to finance more expensive future consumption costs. We expect the risk-free rate to decrease in equilibrium due to such a demand for optimal savings.

We obtain analytical solutions for the agent's optimal consumption and investment choice in the following theorem.

**Theorem 3.2** *The optimal consumption strategy* $c^* \equiv c(0)$ *and the optimal investment strategy* $\pi^* \equiv \pi(0)$ *of the income-disaster-exposed agent are derived analytically as follows*

$$c^* = (\hat{A} + \delta/\gamma)\Big(w + \frac{\xi I}{\beta_1} - \xi I B_\delta^* z^{-\alpha_\delta^*} - IP\Big), \tag{12}$$

$$\begin{aligned}\pi^* = &\frac{1}{\gamma}\sigma^{-1}\theta w + \frac{1}{\gamma}\sigma^{-1}(\theta - \gamma\sigma^I)\Big[\frac{\xi I}{\beta_1} + (\gamma\alpha_\delta^* - 1)\xi I B_\delta^* z^{-\alpha_\delta^*} \\ &- \frac{2\gamma}{||\beta_3||^2}\delta K \frac{\Big(w + \frac{k\xi I}{\beta_1}\Big)^{1-\gamma}}{1-\gamma} \Big/ c^{*-\gamma} + (\gamma\alpha_\delta - 1) \times IP1 + (\gamma\alpha_\delta^* - 1) \times IP2\Big],\end{aligned} \tag{13}$$

[27]It may be reasonable to assume the positive market price of income disaster, i.e., $\hat{\delta} > \delta$ if, and only if $\ln(\hat{\delta}/\delta) > 0$. The negative market price of income disaster would be the case for which an increase in income can be triggered by various reasons, e.g., internet bubble or technology innovation, etc.

*where $\hat{A}$ is defined in (A-26), $\alpha_\delta > 1$ and $-1 < \alpha_\delta^* < 0$ are the two roots of the characteristic equation in (A-28),*

$$\beta_3 = \gamma(\sigma^I)^\top - \theta^\top,$$

*$z$ is a dual variable of financial wealth $w$ and their relation is given in (10), $B_\delta^*$ is a constant to be determined by (A-27), and $IP = IP1 + IP2$ represents the integral parts given by (A-35).*

Without income disaster, i.e. when $\delta = 0$, we have $B_0^* = 0$, $IP = IP1 = IP2 = 0$, and the agent's optimal consumption strategy (12) reduces to that in Merton (1969, 1971):

$$c^* = \hat{A}\Big(w + \frac{\xi I}{\beta_1}\Big),$$

which means that the agent's consumption can be annuitized from her total available financial resources (wealth plus the present value of future income). Furthermore, the MPC out of financial wealth is $\hat{A}$, implying that regardless of wealth levels the agent's optimal consumption to total wealth ratio is maintained at a constant rate.

The classic Merton (1969, 1971) investment rule without income disaster can be also obtained by letting $\delta = 0$:

$$\pi^* = \frac{1}{\gamma}\sigma^{-1}\theta\Big(w + \frac{\xi I}{\beta_1}\Big) - \sigma^{-1}\sigma^I\frac{\xi I}{\beta_1}. \tag{14}$$

The first term on the right hand side of (14) represents the mean-variance asset allocation and the second one represents the demand for hedging (or the intertemporal hedging component) against the aggregate output volatility $\sigma^I$.

Without income disaster, i.e., $\delta = 0$, $IP = IP1 = IP2 = 0$, so that the integral parts $IP$, $IP1$ and $IP2$ have no role in the optimal strategies given in (12) and (13). With income disaster, i.e., $\delta > 0$, the integral parts play important roles in the optimal adjustments of consumption (12) and investment (13), and such extra adjustments reflect the savings motive for precautionary reasons in the event of income disaster.

We now quantitatively identify two different optimal savings motives in the following definition by measuring the wedge between total wealth (financial wealth+human capital) and the sum of consumption and investment: (i) Standard optimal savings and (ii) Income-disaster-induced optimal savings.

**Definition 3.1** *We quantitatively identify two different optimal savings motives as follows.*

*(i) Standard optimal savings*

$$\equiv \Big(w + \frac{\xi I}{\beta_1}\Big) - c(0; B_0^* = 0, \delta = 0) - \pi(0; B_0^* = 0, \delta = 0)$$
$$= \Big(1 - \hat{A} - \frac{1}{\gamma}\sigma^{-1}\theta\Big)\Big(w + \frac{\xi I}{\beta_1}\Big) + \sigma^{-1}\sigma^I \frac{\xi I}{\beta_1}.$$

*(ii) Income-disaster-induced optimal savings*

$$\equiv \Big(w + \frac{\xi I}{\beta_1}\Big) - c(0) - \pi(0)$$
$$= \textit{Standard optimal savings} + \textit{Income-disaster-PS},$$

*where the income-disaster-induced precautionary savings (Income-disaster-PS) are given as follows:*

$$\textit{Income-disaster-PS}$$
$$= -\frac{\delta}{\gamma}\Big(w + \frac{\xi I}{\beta_1}\Big) + (\hat{A} + \delta/\gamma)\Big(\xi I B_\delta^* z^{-\alpha_\delta^*} + IP\Big)$$
$$- \frac{1}{\gamma}\sigma^{-1}(\theta - \gamma\sigma^I)\Big[(\gamma\alpha_\delta^* - 1)\xi I B_\delta^* z^{-\alpha_\delta^*} - \frac{2\gamma}{||\beta_3||^2}\delta K \frac{\Big(w + \frac{k\xi I}{\beta_1}\Big)^{1-\gamma}}{1-\gamma} \Big/ (c^*)^{-\gamma}$$
$$+ (\gamma\alpha_\delta - 1) \times \mathit{IP1} + (\gamma\alpha_\delta^* - 1) \times \mathit{IP2}\Big].$$

The standard optimal savings show that the marginal propensity to save (MPS) out of financial wealth is $1 - \hat{A} - \frac{1}{\gamma}\sigma^{-1}\theta$. This MPS result implies that with respect to one unit increase of wealth, the constant portion of the individual's extra money aside from consumption portion $\hat{A}$ and investment portion $\frac{1}{\gamma}\sigma^{-1}\theta$ is to be optimally put into her riskless savings.

As is fairly standard in consumption/savings models with incomplete markets, there is a large additional precautionary savings term in this model. In addition to the standard optimal savings, the agent cuts down on her consumption by $(\hat{A} + \delta/\gamma) \times IP$ with income disaster as in (12), and such a consumption reduction contributes to increases in the agent's optimal savings especially via Income-disaster-PS in Definition 3.1.

# 4 Income Disaster Model

We consider a simple pure exchange economy in the type of Lucas (1978). The economy is populated by two representative income-endowed traders who have optimized in Section 3 their lifetime consumption over infinite horizon. One is the normal agent ($n$) who is free from income disaster and the other is the income-disaster-exposed agent ($d$) who is subject to a sudden jump shock (driven by a one-time Poisson shock) causing her income being equal to a fraction of its current level.[28] An equilibrium in this two representative-agent economy is defined by the sum of their consumption being equal to the aggregate output (which is proportional to the agents' income endowment) and the sum of stock positions adding up to the agent's financial wealth, resulting in a no-trade equilibrium. In this income disaster model with two representative agents, the price of any Arrow security including the security that insures against the income disaster is such that the net holding of the security by the agents is zero because markets must clear.

The normal agent ($n$) solves the following problem:

$$V_n(w_n, I) = \sup_{(c_n,\pi_n)} E\Big[\int_0^{\tau} e^{-\beta t}\frac{c_n(t)^{1-\gamma}}{1-\gamma}dt\Big],$$

which is subject to the dynamic wealth constraint as follows: $W_n(0) = w_n > -\xi_n I/\beta_1$,

$$dW_n(t) = \{rW_n(t) - c_n(t) + \xi_n I_n(t) + \pi_n(t)^\top(\mu - r\mathbf{1})\}dt + \pi_n(t)^\top\sigma^\top dZ(t),$$

where $I_n(t)$ follows a geometric Brownian motion:

$$dI_n(t) = \mu^I I_n(t)dt + (\sigma^I)^\top I_n(t)dZ(t), \quad I_n(0) = I > 0. \tag{15}$$

The income-disaster-exposed agent ($d$) solves the following problem:

$$V_d(w_d, I) = \sup_{(c_d,\pi_d)} E\Big[\int_0^{\tau} e^{-\beta t}\frac{c_d(t)^{1-\gamma}}{1-\gamma}dt + e^{-\beta\tau}\int_{\tau}^{\infty} e^{-\beta(t-\tau)}\frac{c_d(t)^{1-\gamma}}{1-\gamma}dt\Big],$$

[28] The model with two agents can be rationalized as an extreme simplification to capture the case where one type of agent is more exposed to income disaster than the other. For instance, rare but non-negligible people undergo disastrous income shocks resulting in a dramatic fall in wages at job displacement. Such a wage fall is likely to occur more often in small firms. So, we can consider income disaster encountered by individuals in small firms facing firm closure, and employing forced unemployment risk of individuals working in large firms for consideration of the two agents.

which is subject to the dynamic wealth constraint as follows: $W_d(0) = w_d > -\xi_d I/\beta_1$,

$$dW_d(t) = \{rW_d(t) - c_d(t) + \xi_d I(t) + \pi_d(t)^\top(\mu - r\mathbf{1})\}dt + \pi_d(t)^\top\sigma^\top dZ(t),$$

where $I(t)$ follows (1), with the borrowing limit (4).

Having solved the agents' problems stated above by Section 3, initial wealth levels $w_n$ and $w_d$ are all expressed in terms of initial stock holdings by Merton (1971) and (13):

$$\pi_n = \frac{1}{\gamma}\sigma^{-1}\theta\Big(w_n + \frac{\xi_n I}{\beta_1}\Big)$$

and

$$\pi_d = \frac{1}{\gamma}\sigma^{-1}\theta w_d + \frac{1}{\gamma}\sigma^{-1}(\theta - \gamma\sigma^I)\Big[\frac{\xi_d I}{\beta_1} + (\gamma\alpha_\delta^* - 1)\xi_d I B_\delta^* z^{-\alpha_\delta^*}$$
$$- \frac{2\gamma}{||\beta_3||^2}\delta K \frac{\Big(w + \dfrac{k\xi_d I}{\beta_1}\Big)^{1-\gamma}}{1-\gamma}\Big/ c_d{}^{-\gamma} + (\gamma\alpha_\delta - 1)\times \mathrm{IP1} + (\gamma\alpha_\delta^* - 1)\times \mathrm{IP2}\Big].$$

**Definition 4.1** *An equilibrium is characterized as a collection of* $(r, \mu, \sigma)$ *and optimal strategies* $(c_n(t), c_d(t), \pi_n(t), \pi_d(t))$ *of agent* $n$ *and agent* $d$ *such that the consumption goods, stock and bond markets clear as: for* $t < \tau$,

$$c_n(t) + c_d(t) = I(t),$$
$$\sum_{j=1}^{N}\{\pi_{nj}(t) + \pi_{dj}(t)\} = W_n(t) + W_d(t),$$
$$\nu_{nj}(t) + \nu_{dj}(t) = 1, \quad \textit{for every stock} \;\; j \in \{1, ..., N\},$$

*where the equilibrium has the jump at time* $\tau$ *and the jump size is given by Theorem 3.1 as*

$$\ln\frac{\xi^{\hat{\delta}}(\tau)}{H(\tau-)} = \ln\Big(\frac{\hat{\delta}}{\delta}\Big) - (\hat{\delta} - \delta)\tau, \tag{16}$$

$\nu_{nj}(t)$ *and* $\nu_{dj}(t)$ *represent the proportion of financial wealth invested in the* $j$*-th stock by agent* $n$ *and agent* $d$*, respectively, and* $N$ *is the number of risky stocks.*

We anticipate that in equilibrium the aggregate output $I(t)$ has a jump caused by income disaster at time $\tau$, so its dynamics can be given by (15) for $t < \tau$, but allowing for the jump and its size is determined by (16). In this case, we still have the following relation for $t < \tau$:

$$I(t) = \xi_n I_n(t) + \xi_d I(t) + D(t)$$
$$= (\xi_n + \xi_d) I(t) + D(t)$$
$$= \xi I(t) + D(t),$$

as a result, $D(t) = (1-\xi)I(t)$ for $t < \tau$ demonstrating that dividends are a fraction of aggregate output. Our equilibrium quantities determined in the following theorems reflect before income disaster occurs the effects of such a jump.

**Theorem 4.1** *The equilibrium SDF in the normal-agent economy n is given by*

$$H_n(t) = \frac{1}{\lambda_n} e^{-\beta t} I_n(t)^{-\gamma},$$

*where the constant* $\lambda_n$ *satisfies*

$$E\Big[\int_0^\infty H_n(t)\{(1-\xi_n)I_n(t)\}dt\Big] = w_n.$$

*Before income disaster occurs, the corresponding SDF in the economy d with the agent n and the agent d is*

$$H_d(t) = (\lambda_n^{-1/\gamma} + \lambda_d^{-1/\gamma})^\gamma e^{-\{\beta-(\hat{\delta}-\delta)\}t} I(t)^{-\gamma},$$

*where the constants* $\lambda_n$ *and* $\lambda_d$ *satisfy*

$$E\Big[\int_0^\infty e^{-\hat{\delta}t} H_d(t)\Big(c_n(t) - \xi_n I_n(t) + \hat{\delta} W_n(t)\Big)dt\Big] = w_n,$$
$$E\Big[\int_0^\infty e^{-\hat{\delta}t} H_d(t)\Big(c_d(t) - \xi_d I(t) + \hat{\delta} W_d(t)\Big)\Big] = w_d,$$

$$c_n(t) = \Big(\lambda_n e^{\{\beta-(\hat{\delta}-\delta)\}t} H_d(t)\Big)^{-1/\gamma},$$
$$c_d(t) = \Big(\lambda_d e^{\{\beta-(\hat{\delta}-\delta)\}t} H_d(t)\Big)^{-1/\gamma}.$$

We define the price of the equity market portfolio, $S^{em}$, as the sum of the risky asset prices:

$$S^{em}(t) = \sum_{i=1}^N S_i(t), \quad S^{em}(0) = S^{em},$$

paying out the dividend $D(t) = \sum_{i=1}^{N} d_i(t)$ with $D(0) = D$. The equilibrium price of the equity market portfolio is then given by

$$S^{em}(t) = D(t)E_t\Big[\int_t^{\infty} \frac{H(s)D(s)}{H(t)D(t)} ds\Big],$$

where $H(s)$, $s \geq t$, is the equilibrium SDF given in Theorem 4.1.

**Theorem 4.2** *The equilibrium price of the equity market portfolio in the normal-agent economy $n$ is given by*

$$S_n^{em}(t) = \frac{D(t)}{\beta + (\gamma-1)\mu^I - \frac{1}{2}||\sigma^I||^2\gamma(\gamma-1)}.$$

*Before income disaster occurs, the corresponding equilibrium price in the economy $d$ with with the agent $n$ and the agent $d$ is*

$$S_d^{em}(t) = \frac{D(t)}{\beta - (\hat{\delta}-\delta) + (\gamma-1)\mu^I - \frac{1}{2}||\sigma^I||^2\gamma(\gamma-1)}.$$

Theorem 4.2 shows that the equilibrium equity market portfolio price is the time-$t$ present value of future dividend discounted at different discount rates in the economy $n$ and in the economy $d$. Theorem 4.2 then implies the dynamics of the equilibrium equity market portfolio price as follows

$$dS^{em}(t) = S^{em}(t)\{\mu^I dt + (\sigma^I)^\top dZ(t)\}.$$

**Theorem 4.3** *The equilibrium equity expected return and volatility in the normal-agent economy $n$ are given by*

$$\mu_n^{em} = \beta + \gamma\mu^I - \frac{1}{2}||\sigma^I||^2\gamma(\gamma-1) \quad \text{and} \quad \sigma_n^{em} = \sigma^I,$$

*and the equilibrium risk-free interest rate and Sharpe ratio are given by*

$$r_n = \beta + \gamma\mu^I - \frac{1}{2}\gamma(\gamma+1)||\sigma^I||^2 \quad \text{and} \quad \theta_n = \gamma\sigma^I.$$

*Consequently, the equilibrium equity premium in the economy $n$ is given by*

$$\mu_n^{em} - r_n = \gamma||\sigma^I||^2.$$

*Before income disaster occurs, the corresponding equilibrium quantities in the economy $d$ with the agent $n$ and the agent $d$ are*

$$\mu_d^{em} = \beta - (\hat{\delta} - \delta) + \gamma\mu^I - \frac{1}{2}||\sigma^I||^2\gamma(\gamma - 1) \quad \textit{and} \quad \sigma_d^{em} = \sigma^I,$$

$$r_d = \beta - (\hat{\delta} - \delta) + \gamma\mu^I - \frac{1}{2}\gamma(\gamma + 1)||\sigma^I||^2 \quad \textit{and} \quad \theta_d = \gamma\sigma^I,$$

$$\mu_d^{em} - r_d = \gamma||\sigma^I||^2.$$

Our equilibrium Sharpe ratio and equity premium are not affected by income disaster directly. The Sharpe ratio and equity premium have the usual structure as in the standard asset pricing framework without income disaster. This is expected not surprisingly because within our income disaster model context, the economy is complete and arbitrage-free so that the agents absorb all the aggregate risk, thus coinciding with pure market risk adjustment only especially for the traditional market price of risk (or the Sharpe ratio) $\theta$.

The equilibrium risk-free interest rates in the economy $n$ and in the economy $d$ all increase with the expected consumption growth rate $\mu^I$ and decrease with the consumption growth volatility $||\sigma^I||$. Notice that the square term associated with consumption growth volatility becomes almost zero, because the empirical magnitude of consumption growth volatility is very small. Hence, the interest rate is monotone increasing with risk aversion in the absence of income disaster, i.e., when $\hat{\delta} = \delta$. However, in the presence of income disaster, i.e., when $\hat{\delta} \neq \delta$, the interest rate no longer monotone increases with risk aversion. As far as the positive market price of income disaster is concerned with $\hat{\delta} > \delta$, there are always states of the world in which $r_d$ is lower than $r_n$. The market price of income disaster will increase to compensate for the agent $d$'s risk exposure to income disaster and such risk compensation further increases when the agent $d$ is more risk averse. This risk compensation mechanism leads the interest rate to decrease with risk aversion.

We derive the equilibrium consumption growth dynamics in the economy $d$ with the agent $n$ and the agent $d$.

**Theorem 4.4** *Before income disaster occurs, the equilibrium consumption growth dynamics in the economy d with the agent n and the agent d are given by*

$$\frac{dc_d(t)}{c_d(t)} = \frac{1}{\gamma}\Big(r_d - \beta + \frac{1}{2}\gamma(1+\gamma)||\sigma^I||^2 + \hat{\delta} - \delta\Big)dt + (\sigma^I)^\top dZ(t). \tag{17}$$

Our equilibrium consumption growth dynamics (17) confirms the agent $d$'s three consumption/savings motives in equilibrium: (i) dissavings due to impatience $r_d - \beta$, (ii) precautionary savings due to diffusive-type income risk $\frac{1}{2}\gamma(1+\gamma)||\sigma^I||^2$, and (iii) precautionary savings due to jump-type income disaster $\hat{\delta} - \delta$. Wang (2003) shows that the equilibrium consumption has zero growth rate and hence, it follows a martingale.[29] However, such a martingale property of consumption has been at odds with empirical plausibility in the sense that consumption changes are rather predictable by anticipated future income changes. Therefore, the equilibrium consumption should have its nonzero growth rate. The gap between the theory and empirical reality is known as the excess sensitivity puzzle in consumption (Flavin, 1981).[30] Interestingly, substantial precautionary savings term $\hat{\delta} - \delta$ exists caused by income disaster even in our complete-markets income disaster model, thus leading the equilibrium consumption to have nonzero growth rate helping to resolve the excess sensitivity puzzle.

We derive the agent $d$'s equilibrium marginal propensity to consume (MPC) out of financial wealth $w_d$.

**Theorem 4.5** *The agent d's equilibrium marginal propensity to consume (MPC) out of financial wealth $w_d$ is given by*

$$MPC = \tilde{A}(r_d) + \delta/\gamma, \tag{18}$$

[29] If we assume the equality between $r$ and $\beta$ and turn off the precautionary savings ($||\sigma^I||^2 = 0$) as in the standard literature, we can obtain in the absence of income disaster ($\hat{\delta} = \delta$) exactly the same equilibrium result of Wang (2003).

[30] While the volatility of equilibrium consumption dynamics given in Theorem 4.4 is exactly the same as the consumption growth volatility, $\sigma^I$, and hence income disaster cannot be invoked to rationalize the excess smoothness puzzle raised by Campbell and Deaton (1989).

*where*

$$\tilde{A}(r_d) = r_d + \delta - \mu^I + \gamma||\sigma^I||^2 + \frac{\hat{\delta} - \delta}{\gamma}.$$

Without income disaster, Wang (2003) demonstrates that the equilibrium MPC is merely equal to the risk-free interest rate. This theoretically derived MPC by the risk-free rate is incapable of matching the empirically plausible MPC values. Most macroeconomic models theoretically produce around 4% MPC, whereas the empirically plausible MPC ranges from 20% to 60% (Carroll *et al.*, 2017). More recently, Fisher *et al.* (2019) obtain about 10% MPC which can be regarded as the low end of the MPC range.[31]

With income disaster, the equilibrium MPC given in (18) has extra terms in addition to the risk-free interest rate. The extra terms on the right-hand side of the equation (18) contribute to the income-disaster-perceived effective interest rate. The effective rate decreases with the expected consumption growth rate $\mu^I$ and increases with the income disaster intensity $\delta$, the consumption growth volatility $||\sigma^I||$, and the precautionary savings $\hat{\delta} - \delta$ due to income disaster. In our next quantitative analysis, we will show how large precautionary savings can be generated with income disaster possibility.

# 5 Quantitative Analysis

**Parameter Calibration.** We consider the subjective discount rate $\beta = 4\%$ and the coefficient of relative risk aversion $\gamma = 2$ that are the common values adopted in the literature. For the baseline parameter values for the expected consumption growth rate $\mu^I$ and the volatility of consumption growth rate $\sigma^I$, we use the Robert J. Shiller's real monthly dividend data from 1926 to 2016 and the century-long sample (1891-1994) by Campbell (1999), so that $\mu^I = 1.52\%$, $\sigma^I = 4.06\%$ and $\mu^I = 1.74\%$, $\sigma^I = 3.26\%$, respectively. For numerical illustrations, we do not allow for borrowing against the present value of future

[31] Parker *et al.* (2013) have obtained the empirical MPC estimates ranging from 12% to 30%.

income, i.e., $L = 0$.[32]

The crucial parameter for our income disaster model is the Poisson intensity $\delta$. Income disaster considered in this paper is a permanent income shock that could occur due to forced unemployment and ill-health. Permanent layoffs are empirically plausible especially when individuals work for small firms. Given a positive relation between firm size and credit ratings, if we relate bankruptcy of small firms to the default of speculative-grade firms based on Moody's historical default rates data, the Poisson intensity $\delta$ can be estimated as 4.23% (Jang *et al.*, 2013) or 5.26% (Jang *et al.*, 2019). As to reduced ability to work, e.g., disability, the intensity $\delta$ can be even much greater as 12.05% for good health status and 17.86% for bad health status when matching years of survival in employment depending upon each health status conditional on working at age 57 (Dwyer and Mitchell, 1999). Barro (2006) estimates that income disaster possibility is on average 1.7% per year. Wang *et al.* (2016) have chosen the arrival rate of large discrete (jump) earnings shocks as 5%. In our numerical illustrations, we consider such a reasonable range of values for the Poisson intensity $\delta$ up to 5% in most cases.

The income recovery parameter $k$ in the aftermath of income disaster is set to 40%.[33]

**Implications on Interest Rate.** In our analytical theorem for the general equilibrium quantities (Theorem 4.3) obtained as a result of the analytical solutions for the SDF (Theorem 3.1), we have theoretically identified a decrease in the equilibrium risk-free interest rate by the market price of income disaster measured as the difference $\hat{\delta} - \delta$ between risk-neutral income disaster intensity $\hat{\delta}$ and original (or physical) intensity $\delta$. The income-disaster-exposed agent would be in a more expensive state for future consumption due to the increased SDF drift by (11) (or the increased consumption growth rate as demonstrated

[32]For the effects of borrowing constraints with a range of values for $L$ on the agent's optimal strategies, refer to further numerical results in Section K in Appendix.

[33]In practice, U.S. households have been rescued by a safety net in the aftermath of income disaster (for example, possibly caused by forced unemployment) and recover, at least, 20% of the income that they earned before unemployment (Carroll *et al.*, 2003).

in Theorem 4.4), so the agent is willing to consume less and save more for precautionary reasons. Such a precautionary savings channel should imply a reduction in the equilibrium interest rate with income disaster.

Indeed, in the absence of income disaster ($\delta = 0$), the equilibrium interest rate is 6.55%, but in the presence of income disaster ($\delta > 0$), it drops significantly (Figure 1). This numerical relationship supports our theoretical predictions for interest rates with income disaster as stated above and implies the important discontinuity and dramatic change in the interest rate even when $\delta$ is very small. It turns out that the income-disaster-exposed agent demands a high market risk premium. For instance, the equilibrium interest rate decreases 41.53% (i.e., to 3.83%) as $\delta$ increases from 0 to 2%.

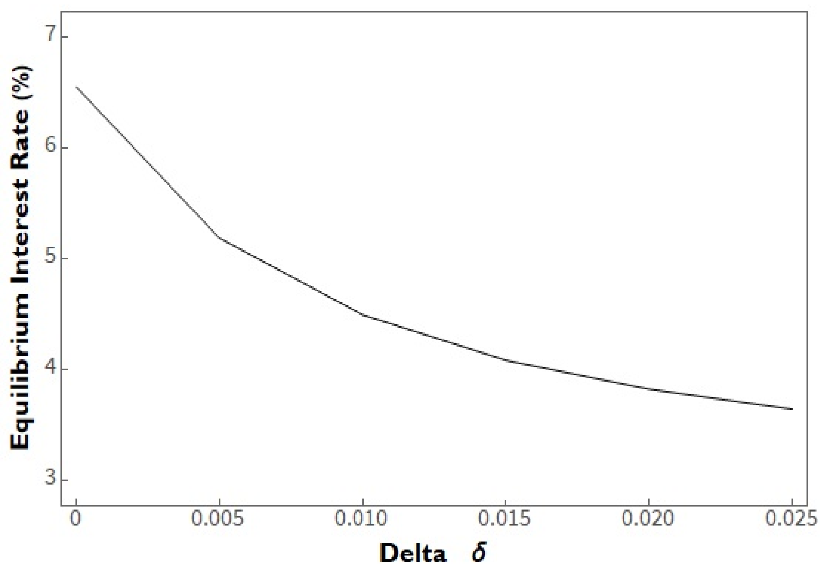


Figure 1: **Equilibrium interest rates**. Parameter Values: $\beta = 4\%$ (subjective discount rate), $\gamma = 2$ (risk aversion), $\mu^I = 1.52\%$ (expected consumption growth rate), $\sigma^I = 4.06\%$ (volatility of consumption growth rate), $\sigma = 20\%$ (stock volatility), $w = 1$ (initial wealth), $I = 1$ (aggregate output), $\xi = 0.5$ (fraction constituting aggregate earnings), $k = 40\%$ (recovery rate), and $L = 0$ (borrowing constraint). For the expected consumption growth rate and volatility of consumption growth rate, $\mu^I$ and $\sigma^I$, we have used the Robert J. Shiller's real monthly dividend data from 1926 to 2016 in "Irrational Exuberance" published by Princeton University Press. Note: In the absence of income disaster ($\delta = 0$), the equilibrium interest rate is 6.55%, but in the presence of income disaster ($\delta > 0$), it drops significantly, implying the important discontinuity and dramatic change in the interest rate even when $\delta$ is small.

Risk aversion $\gamma$ also affects the equilibrium risk-free interest rates (Figure 2). When

$\delta > 0$, high values of $\gamma$ no longer counterfactually generate high risk-free interest rates, so the risk-free rate puzzle (Weil, 1989) is avoided. Rather, an increase in risk aversion can lead to a sizable decrease in risk-free rate in the presence of income disaster.

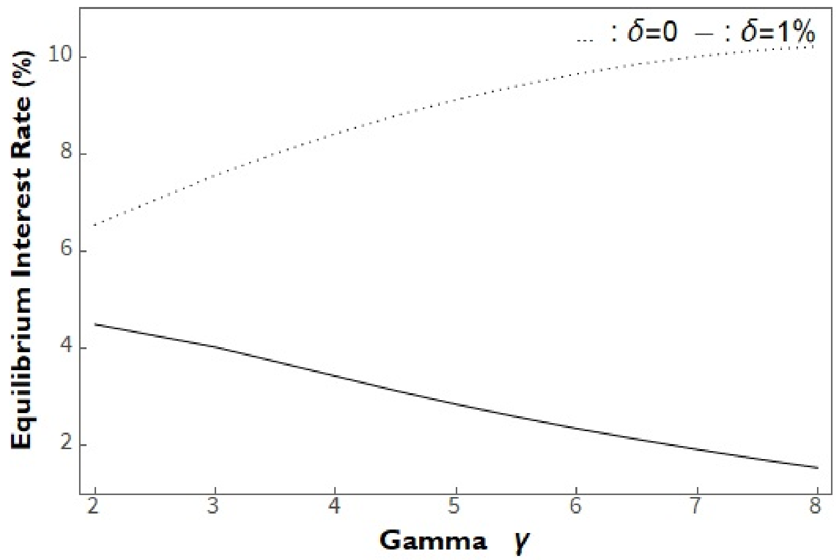


Figure 2: **Equilibrium interest rates**. Parameter Values: $\beta = 4\%$ (subjective discount rate), $\gamma = 2$ (risk aversion), $\mu^I = 1.52\%$ (expected consumption growth rate), $\sigma^I = 4.06\%$ (volatility of consumption growth rate), $\sigma = 20\%$ (stock volatility), $w = 1$ (initial wealth), $I = 1$ (aggregate output), $\xi = 0.5$ (fraction constituting aggregate earnings), $k = 40\%$ (recovery rate), and $L = 0$ (borrowing constraint). Note: When $\delta > 0$, high values of $\gamma$ no longer counterfactually generage high risk-free interest rates, so the risk-free rate puzzle (Weil, 1989) is avoided. Rather, an increase in risk aversion can lead to a decrease in risk-free rate in the presence of income disaster.

The presence of income disaster drives down the risk-free interest rate by stimulating the precautionary savings mechanism, thereby maintaining the risk-free rate low. That is, with income disaster people's demand for precautionary savings is sufficiently strong making her save at a high rate and lowering the equilibrium interest rate significantly.

**Excess Sensitivity Puzzle.** Addressing the excess sensitivity puzzle can be attempted only if we have the nonzero drift of equilibrium consumption growth as theoretically predicted in Theorem 4.4. We have theoretically demonstrated that our equilibrium consumption growth rises with precautionary savings induced by income disaster. We support this theoretical result by numerically showing that the consumption growth rate increases with the income disaster intensity $\delta$ (Figure 3). This numerical result implies that future consumption can be determined by the extent to which how often income will plummet due to income disaster. Intuitively, higher chances of being caught up with income disaster the individual is exposed to, she has to pay more for the same amount of future consumption at the expense of relatively more expensive present consumption. The equilibrium consumption price will, therefore, become more sensitive to changes of future income caused by income disaster.

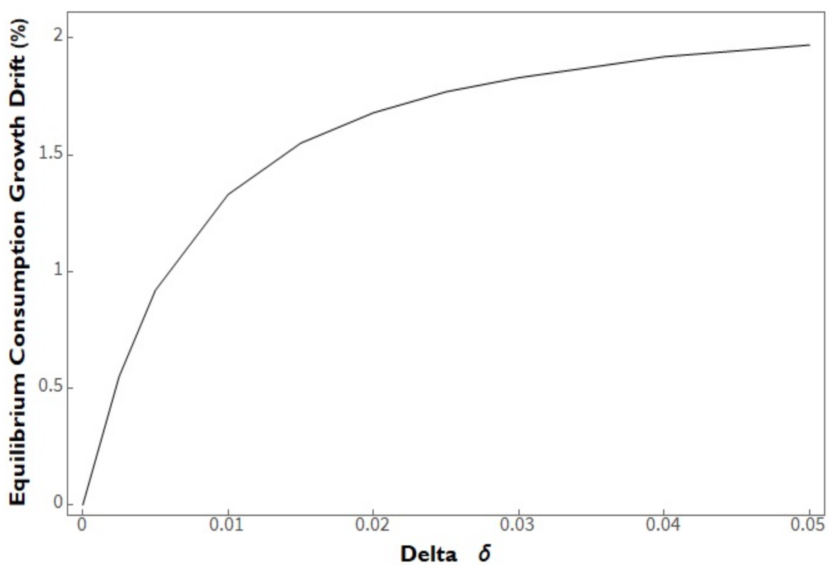


Figure 3: **Equilibrium Consumption Growth Drift**. Parameter Values: $\beta = 4\%$ (subjective discount rate), $\gamma = 3$ (risk aversion), $\mu^I = 1.74\%$ (expected consumption growth rate), $\sigma^I = 3.26\%$ (volatility of consumption growth rate), $\sigma = 18.53\%$ (stock volatility), $w = 1$ (initial wealth), $I = 1$ (aggregate output), $\xi = 0.5$ (fraction constituting aggregate earnings), $k = 40\%$ (recovery rate), and $L = 0$ (borrowing constraint). For the expected consumption growth rate and volatility of consumption growth rate, $\mu^I$ and $\sigma^I$, and stock volatility, $\sigma$, we have used the century-long sample (1891-1994) by Campbell (1999).

**Marginal Propensities to Consume (MPC) Numbers.** In our theoretical analysis on the equilibrium MPC by Theorem 4.5, we have shown that the income-disaster-perceived effective interest rate plays a dominating role in the MPC characterization and in particular, the precautionary savings due to income disaster importantly account for the MPC. In our substantial precautionary savings context with income disaster, we theoretically expect greater consumption responses to changes in wealth.

Without income disaster, the generated MPC values could not reconcile with the empirically plausible MPC ranges (12% to 30% by Parker *et al.*, 2013; 20% to 60% by Carroll *et al.*, 2017; 10% by Fisher *et al.*).[34] This is because without income disaster consumption responses to changes in wealth and income are not very significant for consumption smoothing, as identified in Wang (2003) without income disaster. However, consistent with our theoretical expectation on the MPC behavior with income disaster, our MPC with income disaster can generate even much higher than 8% with just 1% income disaster intensity (Figure 4), which can be quite close to the low end of the MPC range (Fisher *et al.*, 2019).

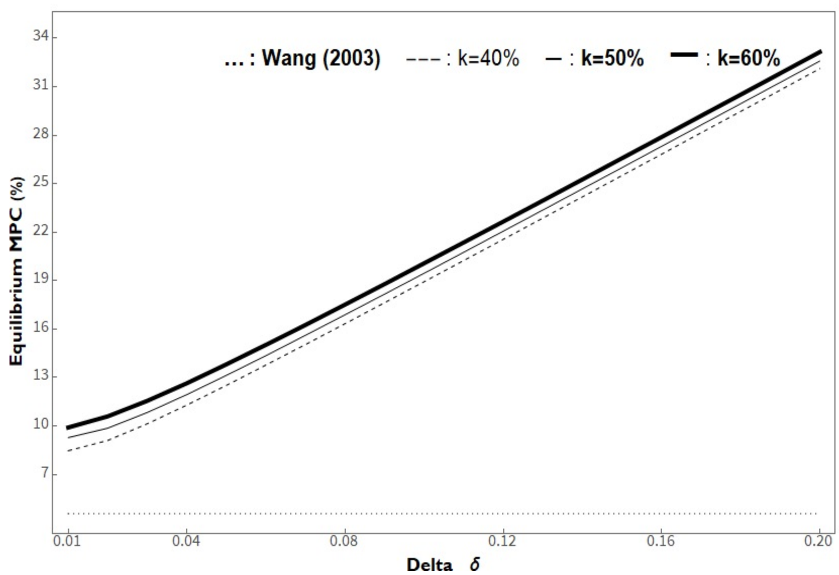


Figure 4: **Equilibrium Marginal Propensity to Consume (MPC)**. Parameter Values: $\beta = 4\%$ (subjective discount rate), $\gamma = 3$ (risk aversion), $\mu^I = 1.74\%$ (expected consumption growth rate), $\sigma^I = 3.26\%$ (volatility of consumption growth rate), $\sigma = 18.53\%$ (stock volatility), $w = 1$ (initial wealth), $I = 1$ (aggregate output), $\xi = 0.5$ (fraction constituting aggregate earnings), and $L = 0$ (borrowing constraint). For the expected consumption growth rate and volatility of consumption growth rate, $\mu^I$ and $\sigma^I$, and stock volatility, $\sigma$, we have used the century-long sample (1891-1994) by Campbell (1999). Wang (2003) is obtained by the result that the MPC equals to the equilibrium interest rate.

[34]Figure 4 demonstrates that the benchmarked Wang (2003)'s MPC without income disaster is around 4% as most macroeconomic models have commonly used, which is quite far from the empirical ranges.

Interestingly, our income recovery parameter $k$ in the event of income disaster turns out to further increase the MPC values. This is because a higher $k$ obviously reduces the amount of the risk premium against income disaster, which results in the smaller market price of income disaster $\ln(\hat{\delta}/\delta) = 0$ (or equivalently, the smaller $\hat{\delta} - \delta$) in Theorem 4.5. This reduced market price result increases the equilibrium interest rate as in the absence of income disaster, which therefore increases the MPC as identified in Theorem 4.5. We then have some policy implications here. In terms of promoting individual consumption especially when hit by income disasters caused by financial crisis, natural catastrophes, pandemics, etc., the effectiveness of government fiscal policy is determined by the extent to which consumption responds promptly to income changes (and wealth changes, accordingly). That is, the explicit efforts by governments should, thus, focus on greater access to (private) insurance and a wide range of government safety nests for individuals to increase their resilience against income disaster. A higher recovery $k$ against income disaster is closely associated with greater consumption responses (MPC) to changes in income and wealth according to our model results.

**Household Savings.** The large additional precautionary savings term induced by income disaster as quantitatively identified in Theorem 3.2 and Definition 3.1 should have implications for households' optimal savings behavior.

In an attempt to better understand the low-consumption-high-savings puzzle, one potential avenue would be to consider an international setting with an extreme simplification as follows. The agent with high disaster exposure represents developing third world countries that depend crucially on the production of commodities whose prices are more susceptible to disaster-like movements. The agent with low disaster exposure represents developed countries that handle most of the manufacturing. In this context, we highlight how different exposure to income disaster in developing and developed countries affects household savings decisions especially with the precautionary savings mechanism in both partial equilibrium and general equilibrium.

We find in partial equilibrium that households in developing countries who encounter

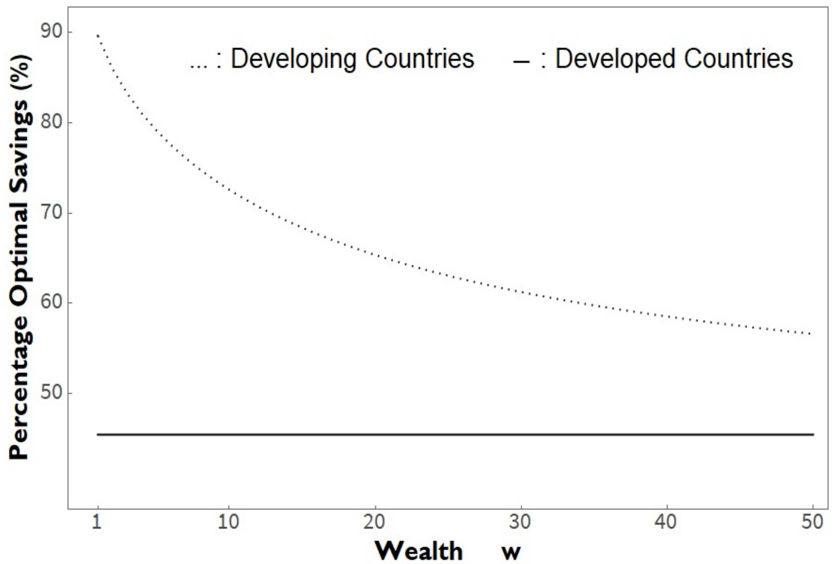


Figure 5: **Precautionary savings in partial equilibrium**. Parameter Values: arameter values: $r = 0.02$ (risk-free rate), $\beta = 0.04$ (subjective discount rate), $\mu = 0.06$ (expected stock return), $\sigma = 0.20$ (stock volatility), $\gamma = 2$ (risk aversion), $\epsilon = 1$ (income), $k = 40\%$ (recovery rate), and $L = 0$ (borrowing constraint). Note: The dotted line represents precautionary savings of households in developing countries who are exposed to 1% income disaster possibility. The solid line represents the savings in developed countries who are not exposed to income disaster.

income disaster tend to maintain at high savings rates for all levels of wealth relative to those in developed countries who are free from income disaster and maintaining at a constant savings rate (Figure 5). The discrepancy of savings rates between developing and developed countries decreases with wealth, because the ability of self-insure against income disaster improves when wealth is large, so precautionary savings decrease as wealth increases.

We find in general equilibrium that households in developing countries tend to consume less and save more than those in developed countries, helping to better explain the low-consumption-high-savings puzzle (Figure 6). Notably, the discrepancies of consumption and savings rates in general equilibrium between developing and developed countries increase with income disaster possibility.

Having theoretically understood both partial and general equilibrium implications of income disaster on consumption and savings decisions of households in developing and

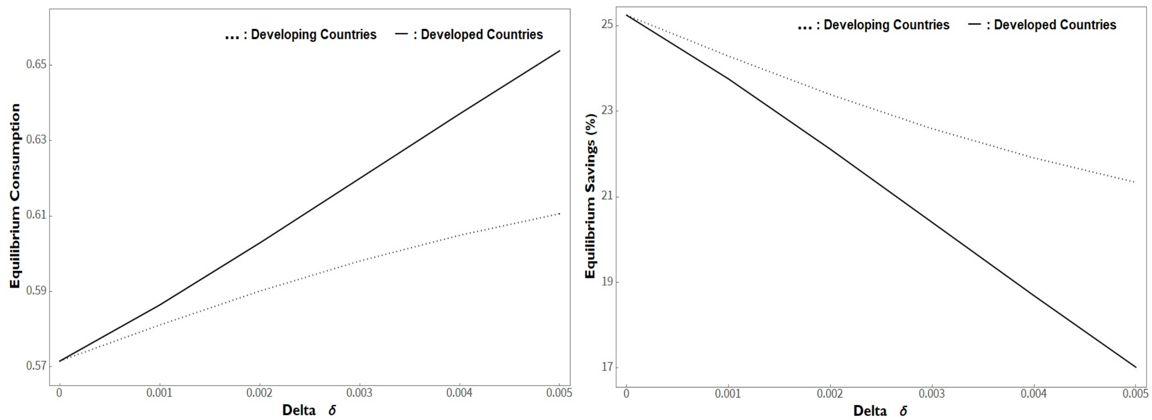


Figure 6: **Precautionary savings in general equilibrium**. Parameter Values: $\beta = 4\%$ (subjective discount rate), $\gamma = 3$ (risk aversion), $\mu^I = 1.74\%$ (expected consumption growth rate), $\sigma^I = 3.26\%$ (volatility of consumption growth rate), $\sigma = 18.53\%$ (stock volatility), $w = 1$ (initial wealth), $I = 1$ (aggregate output), $\xi = 0.5$ (fraction constituting aggregate earnings), $k = 40\%$ (recovery rate), and $L = 0$ (borrowing constraint). For the expected consumption growth rate and volatility of consumption growth rate, $\mu^I$ and $\sigma^I$, and stock volatility, $\sigma$, we have used the century-long sample (1891-1994) by Campbell (1999).

developed countries, we can therefore generate empirically testable implications of income disaster. First, it would be interesting to investigate whether both lower consumption and higher savings rates can be induced indeed in developing countries especially during the challenging times caused by the 2008 global financial crisis and the recent COVID-19 pandemic, when income disaster has been differently evidenced and most of low-income developing countries would struggle to manage income disaster with limited access to insurance and a wide range of government safety nets and social security programs against a large, negative income shock triggered by income disaster. Second, it would be also interesting to investigate how different exposure to income disaster in developing and developed countries affects household consumption and savings choices. Such a differential income disaster exposure may turn out to explain a large part of the savings rate gap between developing and developed countries.

**Asset Pricing Exercise.** Incorporating income disaster, our equilibrium results are matched up with the observed risk-free rate and equity premium. We have tried to match our model with the century-long sample from 1891 to 1994 by Campbell (1999) and the long historical sample from 1871 to 2011 by the website of Robert Shiller (Table 1).[35] The presence of income disaster dramatically improves the model's ability to match asset prices. The standard asset pricing framework by Lucas (1978) without income disaster requires negative values for the subjective discount rate which are not empirically plausible in order to obtain the risk-free rate of 1.74% (1.48%) which is lower than the observed rate of 1.96% (2.8%) from the century-long sample (the long historical sample). While the risk-free rate of 1.96% (2.8%) generated by our framework with income disaster is exactly the same with that observed from the century-long sample (the long historical sample), and thereby requiring empirically plausible parameters as follows: income disaster shock $\delta = 1\%$ ($\delta = 4.5\%$), subjective discount rate $\beta = 1.65\%$ ($\beta = 1\%$), risk aversion $\gamma = 10$ ($\gamma = 9$).[36]

The intuitive interpretation of our improved ability to match the observed asset prices is the income-disaster-exposed agent's optimal savings decision that discourages equity demand so that the equity premium increases. The risk-free rate instead decreases due to the agent's high savings.

Table 2 reports the estimated consumption and return parameter values from four sample periods: 1889-1978 (Lucas, 1994), 1890-1997 (Gomes and Michaelides, 2008), 1929-2009 (Constantinides and Ghosh, 2017), and 1930-2008 (Schmidt, 2016).

Table 3 tests our model with four representative data periods considered by Lucas (Lucas, 1994), GM (Gomes and Michaelides, 2008), CG (Constantinides and Ghosh, 2017), and Schmidt (Schmidt, 2016) who all focus on asset pricing implications of unhedgeable income risk. The second last column of each table reports the number of model parameters used to match asset prices. The details are as follows:

- Lucas (1994) uses 5 model parameters: discount rate $\beta$, risk aversion $\gamma$, income shocks, short sale constraint, borrowing constraint

[35] Refer to http://www.econ.yale.edu/~shiller/data/chap26.xls.

[36] According to Mehra and Prescott (1985), the upper bound of risk aversion is known as 10.

| Estimated consumption and return parameters | 1891-1994 | 1871-2011 |
|---|---|---|
| Expected consumption growth rate $\mu^I$ | 1.74% | 2.3% |
| Consumption volatility $\sigma^I$ | 3.26% | 3.3% |
| Stock Volatility $\sigma$ | 18.53% | 18.2% |
| Risk-free rate $r$ | 1.96% | 2.8% |
| Equity premium $\mu - r$ | 6.26% | 5.2% |

(a) Data

| Required parameters | 1891-1994 | | | | 1871-2011 | | | |
|---|---|---|---|---|---|---|---|---|
| Income Disaster $\delta$ | 0 | 1% | 1% | 10% | 0 | 4.5% | 5.5% | 10% |
| Recovery $k$ | – | 40% | 50% | 60% | – | 40% | 50% | 60% |
| Discount rate $\beta$ | −9.81% | 1.65% | 0.5% | 0.5% | −15.53% | 1% | 0.5% | 0.5% |
| Risk aversion $\gamma$ | 10 | 10 | 10 | 10 | 9 | 9 | 9 | 9 |
| **Model-generated equilibrium quantities** | **1891-1994** | | | | **1871-2011** | | | |
| Risk-free rate | 1.74% | 1.96% | 2.49% | 2.50% | 1.48% | 2.8% | 3.4% | 4.2% |
| Equity premium | 6.04% | 6.04% | 6.04% | 6.04% | 6.0% | 5.4% | 5.4% | 5.4% |

(b) Model results with required parameters

Table 1: Table (a) reports the annualized parameter values for consumption and return for the century-long sample (1891-1994) by Campbell (1999) and the long historical sample (1871-2011) by the website of Robert Shiller (http://www.econ.yale.edu/∼shiller/data/chap26.xls). Table (b) reports a comparison of the model-generated equilibrium results from the model without income disaster ($\delta = 0$) and the model with income disaster ($\delta > 0$).

| Estimated consumption and return parameters | 1889-1978 | 1890-1997 | 1929-2009 | 1930-2008 |
|---|---|---|---|---|
| Expected consumption growth rate $\mu^I$ | 1.8% | 1.7% | 2.0% | 1.93% |
| Consumption volatility $\sigma^I$ | 3.7% | 3.3% | 2.0% | 2.16% |
| Stock Volatility $\sigma$ | 16.7% | 19.81% | 18.7% | 20.28% |
| Risk-free rate $r$ | 1.0% | 1.58% | 0.6% | 0.57% |
| Equity premium $\mu - r$ | 6.0% | 6.74% | 7.0% | 7.09% |

Table 2: Table reports the estimated consumption and return parameter values from sample periods: 1889-1978 (Lucas, 1994), 1890-1997 (Gomes and Michaelides, 2008), 1929-2009 (Constantinides and Ghosh, 2017), and 1930-2008 (Schmidt, 2016).

- Gomes and Michaelides (2008) use 6 model parameters: discount rate $\beta$, risk aversion $\gamma$, two EIS ($\psi$) parameters for two different type agents, deviation of productivity shock, standard deviation of income shock

| Models | Model results: Risk-free rate $r$ | Equity premium $\mu - r$ | Required parameters: Income Disaster $\delta$ | Recovery $k$ | Discount rate $\beta$ | Risk aversion $\gamma$ | Number of parameters | $MSE$ % |
|---|---|---|---|---|---|---|---|---|
| Data | 1.0% | 6.0% | – | – | – | – | – | – |
| Lucas | 9.3% | 0.7% | – | – | 5.0% | 2.5 | 5 | 0.4849 |
| Ours | 3.8% | 1.9% | 5.0% | 40% | 5.0% | 3 | 4 | 0.1233 |
| Ours | 1.0% | 6.0% | 5.0% | 29% | 1.0% | 9.7 | 4 | 0 |

(a) Model results with required parameters (1889-1978)

| Models | Model results: Risk-free rate $r$ | Equity premium $\mu - r$ | Required parameters: Income Disaster $\delta$ | Recovery $k$ | Discount rate $\beta$ | Risk aversion $\gamma$ | EIS $\psi$ | Number of parameters | $MSE$ % |
|---|---|---|---|---|---|---|---|---|---|
| Data | 1.58% | 6.74% | – | – | – | – | – | – | – |
| GM | 2.58% | 3.83% | – | – | 1.0% | 5 | 0.6 | 6 | 0.0473 |
| Ours | 2.20% | 3.27% | 5.0% | 40% | 1.0% | 5 | – | 4 | 0.0621 |
| Ours | 1.58% | 6.74% | 1.9% | 40% | 1.0% | 10.31 | – | 4 | 0 |

(b) Model results with required parameters (1890-1997)

| Models | Model results: Risk-free rate $r$ | Equity premium $\mu - r$ | Required parameters: Income Disaster $\delta$ | Recovery $k$ | Discount rate $\beta$ | Risk aversion $\gamma$ | EIS $\psi$ | Number of parameters | $MSE$ |
|---|---|---|---|---|---|---|---|---|---|
| Data | 0.6% | 7.0% | – | – | – | – | – | – | – |
| CG | 2.5% | 4.9% | – | – | 1.7% | 5.05 | 1.10 | 14 | 0.0401 |
| CG | 4.2% | 3.6% | – | – | 1.3% | 14.7 | – | 14 | 0.1226 |
| Ours | 3.1% | 5.6% | 1.0% | 40% | 1.3% | 15 | – | 4 | 0.0411 |
| Ours | 1.5% | 7.1% | 1.0% | 10% | 1.3% | 19 | – | 4 | 0.0041 |

(c) Model results with required parameters (1929-2009)

| Models | Model results: Risk-free rate $r$ | Equity premium $\mu - r$ | Required parameters: Income Disaster $\delta$ | Recovery $k$ | Discount rate $\beta$ | Risk aversion $\gamma$ | EIS $\psi$ | Number of parameters | $MSE$ |
|---|---|---|---|---|---|---|---|---|---|
| Data | 0.57% | 7.09% | – | – | – | – | – | – | – |
| Schmidt | 0.46% | 6.46% | 8.0% | 49% | 2.55% | 11 | 2 | 21 | 0.0020 |
| Ours | 2.90% | 4.82% | 5.0% | 40% | 2.55% | 11 | – | 4 | 0.0529 |
| Ours | 1.02% | 7.01% | 5.0% | 10% | 2.55% | 16 | – | 4 | 0.0001 |

(d) Model results with required parameters (1930-2008)

Table 3: Table compares our model with four representative general equilibrium models considering unhedgeable income risk by Lucas (Lucas, 1994), GM (Gomes and Michaelides, 2008), CG (Constantinides and Ghosh, 2017), and Schmidt (Schmidt, 2016).

- Constantinides and Ghosh (2017) use 14 model parameters: 3 preference parameters of discount rate $\beta$, risk aversion $\gamma$, EIS ($\psi$), 3 parameters for income shocks, 2 parameters of mean and volatility of aggregate consumption growth, 3 parameters for state variable dynamics, 3 parameters governing aggregate dividend growth dynamics

- Schmidt (2016) uses 21 model parameters including 3 preference parameters of discount rate $\beta$, risk aversion $\gamma$, EIS ($\psi$), and 4 parameters for consumption and income shocks such as income shock intensity $\delta$[37]

- Our model uses 4 model parameters: discount rate $\beta$, risk aversion $\gamma$, income disaster $\delta$, recovery $k$

The last column of each table reports the mean square error (MSE) which is the average squared difference between the observed risk-free rate and equity premium (Data) and the model-generated rates (Model). The last two rows labeled by "Ours" of each table reports the optimized our model results to match asset prices by minimizing the sum of the squared relative errors between the historical rates and our model-generated rates. Overall, we find the incremental contribution to the ability to match asset prices, thus demonstrating the pivotal role of income disaster in matching asset prices.

# 6 General Cases

## 6.1 Repeated income disasters

The income disaster assumption we have considered so far is restrictive by having only one-time income disaster. In this assumption, the uncertainty faced by the agent stems from the random arrival of one-time income disaster. Once income disaster occurs, it will never happen again and hence, the agent goes back to living in the business as usual economy with diversifiable diffusive-type output uncertainty only. In reality, however, agents still anticipate occurrence of income disasters even after income disaster occurs in the past, suffering from sudden shocks causing their income to be equal to a fraction of its current level. It is therefore more realistic to consider an environment where income disasters can happen repeatedly. The rare disaster literature has been thinking about large disaster risks as recurring events that repeat over time (e.g., world wars, the great depression, etc.). As

[37] Refer to Table 3 Summary of Parameters for the Quantitative Model in Schmidt (2016).

a generalization of time-varying rare disasters by Gabaix (2008, 2012) and Wachter (2013), we consider that time-varying income disasters are state dependent as well. This is because income disasters fluctuate in extreme scenarios (e.g., financial distress, market crashes, pandemic outbreak etc.), so arrival rates of Poisson shocks should be state dependent. Given that repeated income disasters with time-varying state-dependent intensity are even worse than the one-time disaster with constant intensity, the economy could generate even more precautionary savings motive caused by income disasters.

For simplicity, we suppose that there are two states: the good state $G$ and the bad state $B$. Over a small time period $(t, t+\Delta t)$, the state switches from the good state $G$ $(B)$ to the bad state $B$ $(G)$ with probability $\phi^G \Delta t$ $(\phi^B \Delta t)$ if the current state is $G$ $(B)$, and stays unaltered with the remaining probability $1-\phi^G \Delta t$ $(1-\phi^B \Delta t)$.

In the presence of time-varying state-dependent income disasters, the aggregate output process $I(t)$ is modeled by a geometric Brownian motion with a general Poisson process as follows:

$$dI(t) = \mu^I I(t-)dt + (\sigma^I)^\top I(t-)dZ(t) - (1-k)I(t-)dN^G(t), \quad I_0 = I > 0,$$

where $N^G(t)$ is the Poisson process with time-varying and state-dependent intensity $\delta^i(t)$, $i \in \{G, B\}$. The dynamics of the intensity $\delta^i(t)$ are assumed to follow in the state $i$, $(i \in \{G, B\})$,

$$d\delta^i(t) = a^i \delta^i(t)dt + b^i \delta^i(t)dZ^\delta(t), \quad \delta^i(0) = \delta^i > 0,$$

where $a^i < 0$ is the intensity's growth rate, $b^i$ is the intensity volatility, and $Z^\delta(t)$ is a standard one-dimensional Brownian motion that is correlated with $Z(t)$ as $dZ(t) \cdot dZ^\delta(t) = (\rho^i)^\top dt$, where $\rho^i = (\rho^i_1, \rho^i_2, ..., \rho^i_n)^\top$, $\rho^i_j \in [-1, 1]$ $(j = 1, 2, ..., n)$. For simplicity, we set $a^i = -\delta^i$. If we further set $b = 0$, the intensity $\delta^i(t)$ then reduces to $\delta^i(t) = \delta^i e^{-\delta^i t}$, which is the probability density function of an exponential distribution with constant intensity $\delta^i$.

The agent's value function with repeated income disasters driven by time-varying state-dependent intensity is given by: in the state $i$, ($i \in \{G, B\}$),

$$V^i(w, I) \equiv \sup_{(c,\pi)} E\Big[\int_0^{\tau^i} e^{-\beta t}\frac{c(t)^{1-\gamma}}{1-\gamma}dt + e^{-\beta\tau^i}V^j(w, I)\Big], \tag{19}$$

subject to (3) and (4), where $\tau^i$ is the first jump time of state changing since the beginning of the state $i$ and $V^j(w, I)$ is the value function in the state $j \neq i$.

The following theorem shows the endogenously determined SDF in an incomplete market caused by time-varying state-dependent income disasters.

**Theorem 6.1** *The SDF in an incomplete market caused by time-varying state-dependent income disasters is endogenously determined as follows: in the state $i$, ($i \in \{G, B\}$),*

$$\xi^{\hat{\delta}^i,\hat{\phi}^i}(t) = \exp\Big\{\ln\Big(\frac{\hat{\delta}^i}{\delta^i}\Big)N^G(t) - (\hat{\delta}^i - \delta^i)t\Big\}\exp\Big\{\ln\Big(\frac{\hat{\phi}^i}{\phi^i}\Big)\boldsymbol{1}_{\{\tau^i=t\}} - (\hat{\phi}^i - \phi^i)t\Big\}H(t),$$

*where $\hat{\delta}^i$ and $\hat{\phi}^i$ are the risk-neutral income disaster intensities given by*

$$\hat{\delta}^i = \delta^i k^{-\gamma}\frac{z(y/k)}{z(y)}$$

*and*

$$\hat{\phi}^i = \phi^i\frac{z_j(y)}{z(y)},$$

*$z(y)$ is the dual variable corresponding to wealth-to-income ratio $y = w/(\xi I)$ by their relation as follows*

$$y + \frac{1}{\beta_1^i} = G_i\big(z(y)\big), \tag{20}$$

$$\beta_1^i = r - \mu^I + (\sigma^I)^\top(\theta + \rho^i b^i),$$

*and $G_i\big(z(y)\big)$ is given in Proposition L.1.*

Theorem 6.1 derive expressions for the endogenous SDF implied by recurrent disasters that is analytically tractable. Using Itô's formula, we obtain before income disaster occurs the following SDF dynamics in the state $i$ ($i \in \{G, B\}$): for $t < \tau^i$,

$$\begin{aligned} d\xi^{\hat{\delta}^i,\hat{\phi}^i}(t) &= -\xi^{\hat{\delta}^i,\hat{\phi}^i}(t)[\{r + (e^{\ln(\hat{\delta}^i/\delta^i)} - 1)\delta + (e^{\ln(\hat{\phi}^i/\phi^i)} - 1)\phi^i\}dt + \theta^\top dZ(t)] \\ &= -\xi^{\hat{\delta}^i,\hat{\phi}^i}(t)[\{r + (\hat{\delta}^i - \delta^i) + (\hat{\phi}^i - \phi^i)\}dt + \theta^\top dZ(t)], \end{aligned}$$

which demonstrates the impact of repeated state-dependent income disasters affecting the SDF with adjustments of the risk-free interest rate by the magnitude of market price of income disaster $\ln(\hat{\delta}^i/\delta^i)$ and the magnitude of market price of state changing $\ln(\hat{\phi}^i/\phi^i)$. The distinct point of repeated income disasters from the one-time income disaster results from the additional adjustment in the SDF dynamics by the market price of state changing. Provided that the market prices of income disaster and state changing are positive, i.e., when $\hat{\delta}^i > \delta^i$ and $\hat{\phi}^i > \phi^i$, the equilibrium consumption price measured as the SDF in Theorem 6.1 becomes even more expensive in the future compared to the case for which the one-time income disaster is considered only. This could therefore give rise to even more precautionary savings motive with repeated income disasters than with the one-time income disaster by giving up much more consumption now to meet future consumption demands.

**Theorem 6.2** *The optimal consumption strategy $c^* \equiv c(0)$ and the optimal investment strategy $\pi^* \equiv \pi(0)$ of the income-disaster-exposed agent in the state $i$, ($i \in \{G, B\}$) are derived analytically as follows*

$$c^* = \Big(\hat{A}_i + \frac{\delta^i + \phi^i}{\gamma}\Big)\Big(w + \frac{\xi I}{\beta_1^i} - \xi I B_i^* z^{-\alpha_i^*} - IP_i\Big),$$

$$\begin{aligned}\pi^* =& \frac{1}{\gamma}\sigma^{-1}(\theta + \rho^i b^i)w \\ &+ \frac{1}{\gamma}\sigma^{-1}(\theta + \rho^i b^i - \gamma\sigma^I)\Big[\frac{\xi I}{\beta_1^i} + (\gamma\alpha_i^* - 1)\xi I B_i^* z^{-\alpha_i^*} \\ &- \frac{2\gamma\phi^i z_j}{||\beta_3^i||^2 z}\Big(w + \frac{\xi I}{\beta_1^i}\Big) - \frac{2\gamma\delta^i k^{1-\gamma}}{||\beta_3^i||^2 z}\xi I v_i\Big(\frac{G_i(z) - 1/\beta_1^i}{k}\Big) \\ &+ (\gamma\alpha_i - 1) \times IP1_i + (\gamma\alpha_i^* - 1) \times IP2_i\Big],\end{aligned}$$

*where*

$$\beta_1^i = r - \mu^I + (\sigma^I)^\top(\theta + \rho^i b^i),$$

$$\beta_3^i = \gamma(\sigma^I)^\top - (\theta + \rho^i b^i)^\top,$$

*$z$ is the dual variable corresponding to wealth-to-income ratio $y = w/(\xi I)$ by their relation (20), $B_i^*$ is a constant to be determined by (A-52), and $IP_i = IP1_i + IP2_i$ represents the integral parts given by (A-57).*

The income-disaster-exposed agent's optimal strategies with repeated income disasters that are fully analytic are useful for quantitatively identifying the income-disaster-induced precautionary savings as similar to Definition 3.1.

**Theorem 6.3** *Before income disaster occurs, the equilibrium equity expected return and volatility in the economy $d$ with the agent $n$ and the agent $d$ are given by*

$$\mu_d^{em,i} = \beta - (\hat{\delta}^i - \delta^i) - (\hat{\phi}^i - \phi^i) + \gamma\mu^I - \frac{1}{2}||\sigma^I||^2\gamma(\gamma-1) \quad \text{and} \quad \sigma_d^{em} = \sigma^I,$$

*and the equilibrium risk-free interest rate and Sharpe ratio are given by*

$$r_d^i = \beta - (\hat{\delta}^i - \delta^i) - (\hat{\phi}^i - \phi^i) + \gamma\mu^I - \frac{1}{2}\gamma(\gamma+1)||\sigma^I||^2 \quad \text{and} \quad \theta_d = \gamma\sigma^I,$$

*where $i \in \{G, B\}$. Consequently, the equilibrium equity premium is given by*

$$\mu_d^{em,i} - r_d^i = \gamma||\sigma^I||^2,$$

*where $i \in \{G, B\}$.*

The negative impact of income disaster that we have analyzed in Theorem 4.3 could be even more reinforced if one considered repeated income disasters instead of the one-time income disaster. Both the market price of income disaster $(\hat{\delta}^i - \delta^i)$ and the market price of state changing $(\hat{\phi}^i - \phi^i)$ contribute to further decrease in the equilibrium risk-free interest rate.

**Theorem 6.4** *The agent $d$'s equilibrium MPC out of financial wealth in the state $i$ ($i \in \{G, B\}$) is given by*

$$MPC^i = r_d^i + \delta^i + \phi^i - \mu^I + \gamma||\sigma^I||^2 + \frac{\hat{\delta}^i - \delta^i}{\gamma} + \frac{\hat{\phi}^i - \phi^i}{\phi^i}.$$

Repeated income disasters are much harder to buffer than the one-time income disaster because the likelihood of future income disaster does change in this general model. So, readiness for income disasters requires even larger amount of savings for consumption smoothing across states and as a result, the MPC rises further with repeated income disasters.

## 6.2 Uninsurable diffusive and continuous income shocks

We have so far developed an income disaster model generating a strong precautionary savings motive rationalizing simultaneously the equity premium puzzle and the risk-free rate puzzle, and matching the empirical MPC. The necessary ingredient here is jumps into labor income. In this section, we clarify why a regular Brownian motion type of income risk with enough volatility that does not consider income disaster cannot produce the same results.

We assume that the aggregate output uncertainty is partially correlated with the market so that it follows

$$dI(t) = \mu^I I(t)dt + \sigma^I I(t)d\tilde{Z}(t),$$

where $\mu^I$ is the constant output mean, $\sigma^I$ is the constant output volatility, and $\tilde{Z}(t)$ is the standard one-dimensional Brownian motion which is correlated with $Z(t)$ in the market as $dZ(t)d\tilde{Z}(t) = \rho^\top dt$, $\rho = (\rho_1, \rho_2, ..., \rho_n)^\top$, $\rho_i \in [-1, 1]$ $(i = 1, 2, ..., N)$. By Levy's theorem, there then exists the one-dimensional Brownian motion $Z^I(t)$ which is independent of $Z(t)$ with the following relation:

$$\tilde{Z}(t) = \rho^\top Z(t) + \sqrt{1 - ||\rho||^2} Z^I(t),$$

where $||\rho||^2 = \sum_{i=1}^N \rho_i^2$. Hence, the aggregate output is now evolved by

$$dI(t) = \mu^I I(t)dt + \sigma^I \rho^\top I(t)dZ(t) + \sigma^I \sqrt{1 - ||\rho||^2} I(t)dZ^I(t),$$

where $Z^I(t)$ can be regarded as uninsurable diffusive and continuous income shocks to the aggregate output.

The following theorem endogenously determines the SDF in an incomplete market caused by uninsurable diffusive and continuous income shocks.

**Theorem 6.5** *The SDF in an incomplete market caused by the uninsurable diffusive and continuous income shocks $Z^I(t)$ is endogenously determined with the following dynamics:*

$$dH^\zeta(t) = -H^\zeta(t)\{rdt + \theta^\top dZ(t) + \zeta dZ^I(t)\}, \quad H^\zeta(0) = 1,$$

*where $\zeta$ is the market price of uninsurable income shocks given by*

$$\zeta = \sigma^I \sqrt{1 - ||\rho||^2} + \sigma^I \sqrt{1 - ||\rho||^2} \frac{G(z) - 1/\tilde{\beta}_1}{zG'(z)},$$

$$\tilde{\beta}_1 = r - \mu^I + \sigma^I \rho^\top \theta,$$

*and $G(z)$ is given in Appendix L.2.*

The SDF with uninsurable diffusive and continuous income shocks given in Theorem 6.5 reduces to the canonical SDF in the complete market when there is no aggregate output uncertainty with $\sigma^I = 0$ and/or when the aggregate output is spanned by the market with $||\rho||^2 = 1$, which results in the zero market price of uninsurable income shocks $\zeta = 0$ so that $H^\zeta(t) = H(t)$. Unlike the income disaster model, uninsurable income shocks do not involve any adjustment of the risk-free interest rate implying that the equilibrium risk-free interest rate is not altered by the income shocks, which is the so-called irrelevance result by Krueger and Lustig (2010).[38] Therefore, income disasters resulting in large, negative jumps in income are crucial for the decrease of the risk-free rate.

# 7 Conclusion

We have developed an analytically tractable continuous-time income disaster model with optimal consumption. The stochastic discount factor in an incomplete market caused by income disaster is endogenously determined, thus explicitly characterizing optimal consumption decisions for two types of agent, one who is exposed to income disaster and another who is not. There is a large additional precautionary savings term in our model which pushes the interest rate down and helps to resolve the risk-free rate puzzle. Interestingly, we find that the equilibrium interest rate is a decreasing function of risk aversion

[38]The uninsurable income shocks driven by Brownian motions that are unspanned by the market do lower the equilibrium risk-free rate especially when agents have preference heterogeneity in risk aversion (Christensen *et al.*, 2012). In contrast to Christensen *et al.* (2012), we consider the simplest possible economic setup with no risk aversion heterogeneity so that unspanned Brownian-type income risks are irrelevant for the equilibrium interest rate, consistent with Krueger and Lustig (2010).

while the equity premium is an increasing function, helping to disentangle the risk-free rate puzzle from the equity premium puzzle. We also find that our substantial precautionary savings mechanism allows large consumption responses to changes in wealth and hence, the model-generated MPCs are quite close to the empirical MPCs. Finally, the precautionary savings motives help to understand the low-consumption-high-savings puzzle.

Many interesting extensions to this model should prove relatively possible. A first important extension would be to allow for housing asset in the model. Approximately two-thirds of U.S. households own their primary residence and for many of them, housing asset is the largest single asset in their household portfolio. So, a sudden shock possibly at times of economic recessions to cause a negative or downward large drop in house prices would have a large impact on the agent's life-cycle consumption and portfolio decisions. Incorporating such housing disaster in the framework of optimal portfolio choice with house prices (e.g., Corradin *et al.*, 2013) is of particular interest on general equilibrium implications because it would allow us to investigate how the properties of returns change over disastrous fluctuations in house prices.

A second extension of the model would be to consider various household heterogeneity observed in the real world (e.g., education, wealth, income, consumption, risk aversion, etc.). For instance, this paper would be greatly generalized if we can introduce risk aversion heterogeneity for asset pricing (e.g., Garleanu and Panageas, 2015) especially before and after income disaster occurrence (e.g., involuntary permanent retirement) while keeping the market incomplete. By doing so, we feature income disaster as an exogenous source of risk aversion heterogeneity, thus investigating the new insights on asset pricing that obtain by risk aversion heterogeneity.

A third extension of the model would be to understand the income disaster channel in habit models. It is well known that individuals are inclined to reluctant to deviate from their past consumption paths especially at times of economic downturns (Cochrane, 2017). Recessions followed by income disaster can be naturally invoked to rationalize habit models and hence, it can be reasonably conjectured that consumption adjustments would be larger responding to income disaster than without habit, and the portfolios geared more towards riskless bond because of the added habit, thus helping to resolve the risk-free rate puzzle well.

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# Appendix

## A A Two-period Model

We establish a simple two-period model with a jump-type income shock. We consider a representative economic agent who aims to attain her optimal consumption and investment strategies over the two periods: period 0 and period 1. The agent dies at the end of period 1 and the probability of her survival until period 1 is $\delta_1$. The objective of the agent is to maximize the following utility function by optimally controlling consumptions $c_0$ and $c_1$ at period 0 and at period 1, respectively:

$$v(c_0) + \delta_2 E[v(c_1)],$$

where $v$ is a strictly increasing, strictly concave real-valued function defined on the set of positive real numbers, $0 < \delta_2 < 1$ is the subjective discount factor, and $E$ denotes the expectation taken at period 0.

There are two tradable financial assets: a riskless bond and a risky stock. The riskless bond pays 1 at period 1 and its price is $\frac{1}{R}$ at period 0, where $R > 0$ is the risk-free interest rate. The price of a share of the risky stock is 1 at period 0 and can be $u$ and $d$ ($u > R > d > 0$) with probabilities $\pi_u$ and $\pi_d = 1 - \pi_u$, respectively, at period 1. The agent obtains aggregate earnings at the rate of $\epsilon$ in each period. There is a jump shock in her earnings that would cause a significant downward jump in earnings from $\epsilon$ to 0 at period 1 with the probability of $p$. The probability distributions of the agent's mortality, the stock price, and the jump shock are assumed to be independent.

The budget constraint during period 1 is described as the following: for $i \in \{u, d\}$,

$$W_{1i} = \begin{cases} Rw_0^B + iw_0^S + \epsilon, & \text{if the income shock does not occur,} \\ Rw_0^B + iw_0^S, & \text{if the income shock occurs,} \end{cases}$$

where $w_0^B$ is the dollar amount of savings invested in the riskless bond during period 0, and $w_0^S$ is the dollar amount of savings invested in the risky stock during period 0.

The optimal consumption strategy $c_{1i}$ at period 1 for $i \in \{u, d\}$ is to consume all of wealth

$W_{1i}$ available at period 1 i.e., $c_{1i} = W_{1i}$. The budget constraint during period 0 is given by

$$W_0 + \epsilon = c_0 + w_0^B + w_0^S,$$

where $c_0$ is the optimal consumption strategy at period 0.

The agent's optimization problem at period 0 is formulated by the following value function:

$$\begin{aligned}
&\max_{(w_0^B, w_0^S)} \Big[ v\Big(W_0 + \epsilon - w_0^B - w_0^S\Big) + \delta_2 E v\Big(W_1\Big)\Big] \\
&= \max_{(w_0^B, w_0^S)} \Big[ v\Big(W_0 + \epsilon - w_0^B - w_0^S\Big) \\
&\qquad + \delta(1-p)\Big\{\pi_u v\Big(Rw_0^B + uw_0^S + \epsilon\Big) + \pi_d v\Big(Rw_0^B + dw_0^S + \epsilon\Big)\Big\} \\
&\qquad + \delta p\Big\{\pi_u v\Big(Rw_0^B + uw_0^S\Big) + \pi_d v\Big(Rw_0^B + dw_0^S\Big)\Big\}\Big],
\end{aligned}$$

where $\delta \equiv \delta_1\delta_2$. The first-order conditions for $w_0^B$ and $w_0^S$ are given by

$$\begin{aligned}
&v'\Big(W_0 + \epsilon - w_0^B - w_0^S\Big) \\
&= (1-p)\delta R\Big\{\pi_u v'\Big(Rw_0^B + uw_0^S + \epsilon\Big) + \pi_d v'\Big(Rw_0^B + dw_0^S + \epsilon\Big)\Big\} \\
&+ p\delta R\Big\{\pi_u v'\Big(Rw_0^B + uw_0^S\Big) + \pi_d v'\Big(Rw_0^B + dw_0^S\Big)\Big\}
\end{aligned}$$

and

$$\begin{aligned}
&v'\Big(W_0 + \epsilon - w_0^B - w_0^S\Big) \\
&= (1-p)\delta\Big\{\pi_u v'\Big(Rw_0^B + uw_0^S + \epsilon\Big)u + \pi_d v'\Big(Rw_0^B + dw_0^S + \epsilon\Big)d\Big\} \\
&+ p\delta\Big\{\pi_u v'\Big(Rw_0^B + uw_0^S\Big)u + \pi_d v'\Big(Rw_0^B + dw_0^S\Big)d\Big\},
\end{aligned}$$

respectively.

To obtain analytically tractable optimal strategies, we assume the simplest possible utility function: $v$ is quadratic and it is given by

$$v(c) = c - \frac{\gamma}{2}c^2,$$

where $\gamma$ is a positive constant. Then the first-order conditions become linear equations and we can derive in closed-form the optimal strategies as the following:

$$\begin{aligned}
w_0^B = &\Big[\delta\{-1 + \gamma(W_0 + \epsilon)\}\{u\pi_u(u - R) - d\pi_d(R - d)\} \\
&\quad + \delta\{(1-p)\gamma\epsilon - 1\}\{(u\pi_u + d\pi_d - R) - \delta R\pi_u\pi_d(u-d)^2\}\Big] \\
&\quad \Big/ \gamma\Big[(1 + \delta R^2)\{1 + \delta(u^2\pi_u + d^2\pi_d)\} - \{1 + \delta R(u\pi_u + d\pi_d)\}^2\Big],
\end{aligned}$$

$$w_0^S = \Big[\delta(u\pi_u + d\pi_d - R)\Big(R\{-1+\gamma(W_0+\epsilon)\} + \{(1-p)\gamma\epsilon - 1\}\Big)\Big]$$
$$\Big/\gamma\Big[\{1+\delta R(u\pi_u + d\pi_d)\}^2 - \{1+\delta R^2(\pi_u+\pi_d)\}\{1+\delta(u^2\pi_u + d^2\pi_d)\}\Big].$$

Notice that the premium term $u\pi_u + d\pi_d - R$ on the risky stock can be reasonably assumed to be positive. Then,

$$\frac{w_0^S}{\partial p} < 0.$$

This arguably states that the income shock reduces the dollar amount of savings invested in the risky stock.

However, we need to investigate whether this result does change relying on the assumptions under which the utility function is quadratic and only two periods rather than multi periods are considered. If we relax those assumptions by considering the well-known utility functions (the constant absolute or relative risk aversion utility function) or multi-period settings, to our best knowledge, the first-order conditions obtained when deriving optimal strategies turn out to be highly non-linear, so it is a considerable challenge to solve the problem analytically or even numerically.

Instead of the discrete time two-period model with the quadratic utility function, we will now develop a tractable continuous-time model with the constant relative risk aversion (CRRA) utility function, where all the optimal strategies and general equilibrium quantities are analytically tractable and derived in closed-form.

# B Human Capital Value Calculations

Following Koo (1998), the CEPV can be defined as

$$\text{CEPV} = E\Big[\int_0^\infty H(t)\xi I(t)dt\Big],$$

where $H(t)$ is the standard SDF that is given in Theorem 3.1 and $\xi I(t)$ is the agent's lifetime labor income with the following geometric Brownian motion dynamics of $I(t)$:

$$dI(t) = \mu^I I(t)dt + (\sigma^I)^\top I(t)dZ(t), \quad I(0) = I > 0.$$

The CEPV is then given by

$$\text{CEPV} = \frac{\xi I}{\beta_1},$$

where $\beta_1$ is given in (5), which serves as the human capital with which the agent's borrowing limit is bounded from below as in (4).

# C Derivation Details behind Problem (8)

Following Merton (1969, 1971), consider the agent's problem without the Poisson jump

$$V^A(w, I) \equiv \sup_{(c,\pi)} E\Big[\int_0^\infty e^{-\beta t}\frac{c(t)^{1-\gamma}}{1-\gamma}dt\Big], \tag{A-1}$$

subject to

$$dW(t) = \{rW(t) - c(t) + \xi I(t) + \pi(t)^\top(\mu - r\mathbf{1})\}dt + \pi(t)^\top\sigma^\top dZ(t), \quad W(0) = w > -\xi I/\beta_1,$$

$$W(t) > -\frac{\xi I(t)}{\beta_1}, \quad \text{for all } t \geq 0.$$

The problem (A-1) is solved analytically as

$$V^A(w, I) = K\frac{\{W(t) + \xi I(t)/\beta_1\}^{1-\gamma}}{1-\gamma},$$

where

$$K = \Big[\frac{\gamma - 1}{\gamma}\Big(r + \frac{||\theta||^2}{2\gamma}\Big) + \frac{\beta}{\gamma}\Big]^{-\gamma}. \tag{A-2}$$

Now the agent's new problem with the Poisson jump is given by

$$V(w, I) \equiv \sup_{(c,\pi)} E\Big[\int_0^\tau e^{-\beta t}\frac{c(t)^{1-\gamma}}{1-\gamma}dt + e^{-\beta\tau}V^A(w, kI)\Big],$$

where $\tau$ represents the arrival of the Poisson shock. After integrating out the Poisson intensity $\delta$, the problem stated above becomes exactly the same as Problem (8). The derivation details are as follows.

$$
\begin{aligned}
V(w,I) &= \sup_{(c,\pi)} E\Big[\int_0^{\tau} e^{-\beta t}\frac{c(t)^{1-\gamma}}{1-\gamma}dt + e^{-\beta\tau}V^A(W(\tau),kI(\tau))\Big] \\
&= \sup_{(c,\pi)} E\Big[\int_0^{\tau} e^{-\beta t}\frac{c(t)^{1-\gamma}}{1-\gamma}dt + e^{-\beta\tau}K\frac{\{W(\tau)+k\xi I(\tau)/\beta_1\}^{1-\gamma}}{1-\gamma}\Big] \\
&= \sup_{(c,\pi)} E\Big[\int_0^{\infty} \delta e^{-\delta s}\int_0^{s} e^{-\beta t}\frac{c(t)^{1-\gamma}}{1-\gamma}dtds \\
&\qquad + \int_0^{\infty} \delta e^{-\delta t}e^{-\beta t}K\frac{\{W(t)+k\xi I(t)/\beta_1\}^{1-\gamma}}{1-\gamma}dt\Big] \\
&= \sup_{(c,\pi)} E\Big[\int_0^{\infty} e^{-\beta t}\frac{c(t)^{1-\gamma}}{1-\gamma}\int_t^{\infty} \delta e^{-\delta s}dsdt \\
&\qquad + \int_0^{\infty} e^{-(\beta+\delta)}\delta K\frac{\{W(t)+k\xi I(t)/\beta_1\}^{1-\gamma}}{1-\gamma}dt\Big] \\
&= \sup_{(c,\pi)} E\Big[\int_0^{\infty} e^{-(\beta+\delta)t}\frac{c(t)^{1-\gamma}}{1-\gamma}dt + \int_0^{\infty} e^{-(\beta+\delta)}\delta K\frac{\{W(t)+k\xi I(t)/\beta_1\}^{1-\gamma}}{1-\gamma}dt\Big] \\
&= \sup_{(c,\pi)} E\Big[\int_0^{\infty} e^{-(\beta+\delta)t}\Big(\frac{c(t)^{1-\gamma}}{1-\gamma} + \delta K\frac{\{W(t)+k\xi I(t)/\beta_1\}^{1-\gamma}}{1-\gamma}\Big)dt\Big].
\end{aligned}
$$

# D Proof of Theorem 3.1

We now explicitly characterize stochastic discount factors (SDFs) in our incomplete market caused by income disaster. The following result is the equivalent result in Blanchet-Scalliet *et al.* (2005) to just consider constant intensity of the random time $\tau$. The following result is a straightforward extension of the result in Blanchet-Scalliet *et al.* (2005) and is, thus, given without proof.

**Lemma D.1** *The SDFs are given by*

$$
\xi^{\hat{\delta}}(t) \equiv \exp\Big\{\ln\Big(\frac{\hat{\delta}}{\delta}\Big)N(t) - (\hat{\delta}-\delta)t\Big\}H(t), \tag{A-3}
$$

*where $H(t)$ is the standard SDF in complete markets under no arbitrage and its dynamics are given by*

$$
dH(t) = -H(t)\{rdt + \theta^{\top}dZ(t)\}, \quad H(0)=1,
$$

*and $\hat{\delta}$ is the risk-neutral intensity to be determined.*

We use $\xi^{\hat{\delta}}(t)$ to change the measure. We define

$$\tilde{P}(A) \equiv \int_A e^{rt}\xi^{\hat{\delta}}(t,\omega)dP(\omega) \quad \text{for all} \quad A \in \mathcal{G}. \tag{A-4}$$

In the following lemma, we can establish the Brownian motion process under the probability measure $\tilde{P}$.

**Lemma D.2** *Under the probability measure $\tilde{P}$, the process*

$$\tilde{Z}(t) \equiv \theta dt + Z(t)$$

*is a Brownian motion and $N(t)$ is a Poisson process with intensity $\hat{\delta}$.*

**Proof.** We need to show that $\tilde{Z}(t)$ and $N(t)$ have the correct joint moment-generating function under $\tilde{P}$. So, we must show

$$\tilde{E}[e^{u_1\tilde{Z}(t)+u_2N(t)}] = e^{\frac{1}{2}u_1^2t}\exp\{\hat{\delta}t(e^{u_2}-1)\},$$

where $e^{\frac{1}{2}u_1^2t}$ is the moment-generating function for a normal random variable with mean zero and variance $t$ and $\exp^{\hat{\delta}t(e^{u_2}-1)}$ is the moment-generating function for a Poisson process with intensity $\hat{\delta}$. We can obtain the following independence-based computation:

$$\begin{aligned}
&\tilde{E}[e^{u_1\tilde{Z}(t)+u_2N(t)}] \\
&= E[e^{u_1\tilde{Z}(t)}H(t)]E\Big[e^{u_2N(t)}\exp\Big\{\ln\Big(\frac{\hat{\delta}}{\delta}\Big)N(t)-(\hat{\delta}-\delta)t\Big\}\Big] \\
&= e^{\frac{1}{2}u_1^2t}e^{(\delta-\hat{\delta})t}E\Big[\exp\Big\{\Big(u_2+\ln\frac{\hat{\delta}}{\delta}\Big)N(t)\Big\}\Big] \\
&= e^{\frac{1}{2}u_1^2t}e^{(\delta-\hat{\delta})t}\exp\Big\{\delta t\Big(e^{u_2+\ln(\hat{\delta}/\delta)}-1\Big)\Big\} \\
&= e^{\frac{1}{2}u_1^2t}\exp\{\hat{\delta}t(e^{u_2}-1)\},
\end{aligned}$$

which completes the proof. **Q.E.D.**

For a fixed $\hat{\delta}$, we provide a lemma to convert the dynamic wealth constraint in (3) into the static wealth constraint as follows.

**Lemma D.3** *For a fixed $\hat{\delta}$, the dynamic wealth constraint (3) can be converted into the following static wealth constraint:*

$$E\Big[\int_0^\infty e^{-\hat{\delta}t}H(t)\Big(c(t)-\xi I(t)+\hat{\delta}W(t)\Big)dt\Big] \le w.$$

**Proof.** By applying Itô's formula to $d\big(e^{-rt}W(t)\big)$ yields

$$d\big(e^{-rt}W(t)\big) = -e^{-rt}\{c(t) - \xi I(t)\}dt + e^{-rt}\pi(t)^\top d\tilde{Z}(t), \tag{A-5}$$

where $\tilde{Z}$ is the Brownian motion process by Lemma D.2 under the new martingale measure (A-4) with respect to the SDF $\xi^{\hat{\delta}}(t)$ given in Lemma D.1. Integrating the both sides of (A-5) from 0 to $\tau$,

$$\int_0^\tau e^{-rt}\Big(c(t) - \xi I(t)\Big)dt + e^{-r\tau}W(\tau) = w + \int_0^\tau e^{-rt}\pi(t)^\top d\tilde{Z}(t).$$

Taking expectation $\tilde{E}$ under the new martingale measure,

$$\tilde{E}\Big[\int_0^\tau e^{-rt}\Big(c(t) - \xi I(t)\Big)dt + e^{-r\tau}W(\tau)\Big] \le w.$$

Changing the martingale measure into the physical measure using the relationship (A-4),

$$E\Big[\int_0^\tau \xi^{\hat{\delta}}(t)\Big(c(t) - \xi I(t)\Big)dt + \xi^{\hat{\delta}}(\tau)W(\tau)\Big] \le w.$$

Integrating out the Poisson intensity $\delta$ with respect to $\tau$ using the conditional expectation completes the proof of the lemma. **Q.E.D.**

**Remark.** Lemma D.3 implies that the unique $\hat{\delta}$ to be found (resulting in the unique SDF by Lemma D.1) should lead

$$\int_0^\tau e^{-rt}\Big(c(t) - \xi I(t)\Big)dt + e^{-r\tau}W(\tau)$$

to be a martingale under the probability measure $\tilde{P}$ by (A-4) so that the following risk-neutral pricing formula for the agent's wealth is obtained:

$$w = \tilde{E}\Big[\int_0^\tau e^{-rt}\Big(c(t) - \xi I(t)\Big)dt + e^{-r\tau}W(\tau)\Big],$$

which guarantees that the agent's optimal wealth process $W(t)$ is self-financed by the agent's optimal consumption choice $c(t)$.

With the help of Lemma D.3, the original dynamic problem (8) can be converted into the following static problem:

$$V(w, I) = \sup_{(c,W)} E\Big[\int_0^\infty e^{-(\beta+\delta)t}\Big(\frac{c(t)^{1-\gamma}}{1-\gamma} + \delta K\frac{\{W(t) + k\xi I(t)/\beta_1\}^{1-\gamma}}{1-\gamma}\Big)dt\Big], \tag{A-6}$$

subject to the following static budget constraint for a fixed $\hat{\delta}$:

$$E\Big[\int_0^\infty e^{-\hat{\delta}t}H(t)\Big(c(t)-\xi I(t)+\hat{\delta}W(t)\Big)dt\Big]\leq w.$$

We then introduce a lemma to reformulate the problem (A-6).

**Lemma D.4** *The static optimization problem (A-6) can be reformulated as*

$$V(w,I)=\inf_{(\lambda,\hat{\delta})}\{J^{\hat{\delta}}(\lambda,I)+\lambda w\}=\inf_{\lambda}\{\inf_{\hat{\delta}}J^{\hat{\delta}}(\lambda,I)+\lambda w\}\equiv\inf_{\lambda}\{J(\lambda,I)+\lambda w\},\tag{A-7}$$

*where the indirect value function* $J^{\hat{\delta}}(\lambda,I)$ *is given by*

$$\begin{aligned}J^{\hat{\delta}}(\lambda,I)&=(\xi I)^{1-\gamma}\tilde{E}\Big[\int_0^\infty e^{-(\beta_2+\delta)t}\Big\{\frac{\gamma}{1-\gamma}\Big(1+(\delta K)^{1/\gamma}\hat{\delta}^{1-1/\gamma}\Big)\Gamma^{\hat{\delta}}(t)^{1-1/\gamma}+\Big(1+\frac{\hat{\delta}k}{\beta_1}\Big)\Gamma^{\hat{\delta}}(t)\Big\}dt\Big]\\&\equiv(\xi I)^{1-\gamma}\varphi_{\hat{\delta}}(z)\end{aligned}\tag{A-8}$$

*with* $z=\lambda(\xi I)^\gamma$*, where* $\tilde{E}$ *is the expectation under the new probability measure defined as*

$$\tilde{P}(A)\equiv\int_A\exp\Big(-\frac{1}{2}(1-\gamma)^2||\sigma^I||^2(t,\omega)+(1-\gamma)(\sigma^I)^\top Z(t,\omega)\Big)dP(\omega)\quad\textit{for all}\quad A\in\mathcal{G}$$

*with the new Brownian motion process* $\tilde{Z}$ *given by*

$$\tilde{Z}(t)=-(1-\gamma)\sigma^I t+Z(t),$$

$\Gamma^{\hat{\delta}}(t)$ *is a new state variable defined by*

$$\Gamma^{\hat{\delta}}(t)\equiv\lambda e^{(\beta+\delta-\hat{\delta})t}H(t)(\xi I(t))^\gamma,$$

*and its dynamics are given by*

$$d\Gamma^{\hat{\delta}}(t)=\Gamma^{\hat{\delta}}(t)\{-(\beta_1^{\hat{\delta}}-\beta_2)dt+\beta_3 d\tilde{Z}(t)\}$$

*with*

$$\begin{aligned}\beta_1^{\hat{\delta}}&\equiv r-\mu^I+(\sigma^I)^\top\theta+\hat{\delta}-\delta,\\\beta_2&=\beta-(1-\gamma)\mu^I+\frac{1}{2}\gamma(1-\gamma)||\sigma^I||^2,\\\beta_3&=\gamma(\sigma^I)^\top-\theta^\top.\end{aligned}\tag{A-9}$$

**Proof.** Using the standard Lagrangian approach, we can construct the indirect value function, $J^{\hat{\delta}}(\lambda, I)$, and it is given by

$$J^{\hat{\delta}}(\lambda, I) \equiv \sup_{(c,W)} E\Big[\int_0^\infty e^{-(\beta+\delta)t}\Big(\frac{c(t)^{1-\gamma}}{1-\gamma} + \delta K \frac{\{W(t)+k\xi I(t)/\beta_1\}^{1-\gamma}}{1-\gamma}\Big)dt\Big] \\ -\lambda E\Big[\int_0^\infty e^{-\hat{\delta}t} H(t)\Big(c(t) - \xi I(t) + \hat{\delta} W(t)\Big)dt\Big]. \tag{A-10}$$

Applying the first-order conditions for consumption $c(t)$ and wealth $W(t)$ gives rise to

$$c(t) = \Big(\lambda e^{(\beta+\delta-\hat{\delta})t} H(t)\Big)^{-1/\gamma}, \\ W(t) = \Big(\lambda e^{(\beta+\delta-\hat{\delta})t} H(t)\Big)^{-1/\gamma}\Big(\frac{\hat{\delta}}{\delta}\Big)^{-1/\gamma} K^{1/\gamma} - k\xi I(t)/\beta_1. \tag{A-11}$$

The indirect value function in (A-10) can be rewritten when the above first-order conditions for consumption and wealth are substituted in:

$$J^{\hat{\delta}}(\lambda, I) = E\Big[\int_0^\infty e^{-(\beta+\delta)t}\Big\{\frac{\gamma}{1-\gamma}\Big(\lambda e^{(\beta+\delta-\hat{\delta})t} H(t)\Big)^{1-1/\gamma} \\ + \frac{\gamma}{1-\gamma}(\delta K)^{1/\gamma}\hat{\delta}^{1-1/\gamma}\Big(\lambda e^{(\beta+\delta-\hat{\delta})t} H(t)\Big)^{1-1/\gamma} \\ + \Big(1 + \frac{\hat{\delta}k}{\beta_1}\Big)\Big(\lambda e^{(\beta+\delta-\hat{\delta})t} H(t)\xi I(t)\Big)\Big\}dt\Big]. \tag{A-12}$$

We introduce a new state variable to reformulate the indirect value function in (A-12). Specifically,

$$\Gamma^{\hat{\delta}}(t) \equiv \lambda e^{(\beta+\delta-\hat{\delta})t} H(t)(\xi I(t))^{\gamma}.$$

The indirect value function in (A-10) can be reformulated as the function of $\Gamma^{\hat{\delta}}(t)$:

$$J^{\hat{\delta}}(\lambda, I) = E\Big[\int_0^\infty (\xi I(t))^{1-\gamma} e^{-(\beta+\delta)t}\Big\{\frac{\gamma}{1-\gamma}\Big(\Gamma^{\hat{\delta}}(t)^{1-1/\gamma} + (\delta K)^{1/\gamma}\hat{\delta}^{1-1/\gamma}\Gamma^{\hat{\delta}}(t)^{1-1/\gamma}\Big) \\ + \Big(1 + \frac{\hat{\delta}k}{\beta_1}\Big)\Gamma^{\hat{\delta}}(t)\Big\}dt\Big].$$

By Girsanov's theorem, the new probability measure can be defined by

$$\tilde{P}(A) \equiv \int_A \exp\Big(-\frac{1}{2}(1-\gamma)^2 ||\sigma^I||^2(t,\omega) + (1-\gamma)(\sigma^I)^\top Z(t,\omega)\Big)dP(\omega) \quad \text{for all} \quad A \in \mathcal{G}$$

and the new Brownian motion process $\tilde{Z}$ is given by

$$\tilde{Z}(t) = -(1-\gamma)\sigma^I dt + Z(t).$$

The dynamics of the new state variable follow

$$d\Gamma^{\hat{\delta}}(t) = \Gamma^{\hat{\delta}}(t)\{-(\beta_1^{\hat{\delta}} - \beta_2)dt + \beta_3 d\tilde{Z}(t)\},$$

where $\beta_1^{\hat{\delta}}$, $\beta_2$, and $\beta_3$ are the constants given in the theorem. As a result, the indirect value function is given by

$$J^{\hat{\delta}}(\lambda, I) = (\xi I)^{1-\gamma}\tilde{E}\Big[\int_0^\infty e^{-(\beta_2+\delta)t}\Big\{\frac{\gamma}{1-\gamma}\Big(1+(\delta K)^{1/\gamma}\hat{\delta}^{1-1/\gamma}\Big)\Gamma^{\hat{\delta}}(t)^{1-1/\gamma} + \Big(1+\frac{\hat{\delta}k}{\beta_1}\Big)\Gamma^{\hat{\delta}}(t)\Big\}dt\Big]$$
$$\equiv (\xi I)^{1-\gamma}\varphi_{\hat{\delta}}(z),$$

where $z = \lambda(\xi I)^\gamma$. Following Karatzas *et al.* (1991), the original dynamic problem (8) or equivalently, the static problem (A-6) essentially derives from the indirect value function in (A-10) by (A-7), from which the proof is completed. **Q.E.D.**

We now determine $\hat{\delta}$ uniquely and explicitly, which completes the proof of the theorem.

**Lemma D.5** *The risk-neutral intensity $\hat{\delta}$ is determined uniquely by*

$$\hat{\delta} = \Big(\frac{w}{\xi I} + \frac{k}{\beta_1}\Big)^{-\gamma}\frac{\delta K}{z}, \tag{A-13}$$

*where $z$ is the corresponding marginal value of $w$ by their relation as follows:*

$$G(z) = \Big(w + \frac{\xi I}{\beta_1}\Big)\Big/(\xi I),$$

*which represents the total wealth to income ratio to be optimally determined by solving the following non-linear differential equation:*

$$-\frac{1}{2}||\beta_3||^2 z^2 G''(z) - (||\beta_3||^2 + \beta_2 + \delta - \beta_1)zG'(z)$$
$$+ \beta_1 G(z) + \delta K\Big(G(z) - \frac{1}{\beta_1} + \frac{k}{\beta_1}\Big)^{-\gamma} G'(z) = z^{-1/\gamma}, \quad 0 < z < \overline{z}, \tag{A-14}$$

*where $\overline{z}$ is a constant to be determined by the boundary conditions as follows:*

$$G(\overline{z}) = -\frac{L}{\xi I} + \frac{1}{\beta_1} \quad \textit{and} \quad G'(\overline{z}) = 0.$$

**Proof.** The function $\varphi_{\hat{\delta}}(z)$ in (A-8) should satisfy by Feynman-Kac's formula the following non-linear ordinary differential equation:

$$\inf_{\hat{\delta}}\Big[\frac{1}{2}||\beta_3||^2 z^2\varphi''_{\hat{\delta}}(z) - (\beta_1^{\hat{\delta}} - \beta_2)z\varphi'_{\hat{\delta}}(z) - (\beta_2+\delta)\varphi_{\hat{\delta}}(z)$$
$$+ \frac{\gamma}{1-\gamma}\Big(1+(\delta K)^{1/\gamma}\hat{\delta}^{1-1/\gamma}\Big)z^{1-1/\gamma} + \Big(1+\frac{\hat{\delta}k}{\beta_1}\Big)z\Big] = 0, \quad 0 < z < \overline{z}, \tag{A-15}$$

where

$$\beta_1^{\hat{\delta}} = r - \mu^I + (\sigma^I)^\top \theta + \hat{\delta} - \delta,$$
$$\beta_2 = \beta - \mu^I(1-\gamma) + \frac{1}{2}\gamma(1-\gamma)||\sigma^I||^2,$$
$$\beta_3 = \gamma(\sigma^I)^\top - \theta^\top,$$

and $\overline{z}$ is to be determined according to the boundary conditions (or the value matching and smooth pasting conditions) given by

$$\varphi'_{\hat{\delta}}(\overline{z}) = \frac{L}{\xi I}, \quad \varphi''_{\hat{\delta}}(\overline{z}) = 0.$$

Note that the technical details behind the boundary conditions stated above are a result of the fact that by (A-7),

$$w = -\xi I \varphi'_{\hat{\delta}}(z)$$

so that financial wealth $w$ goes down to its lower bound $-L$ given in (4) as $z$ approaches $\overline{z}$ and the fact that the local minimum of $\varphi'_{\hat{\delta}}(z)$ is achieved at $\overline{z}$ given that $\varphi'_{\hat{\delta}}(z)$ is a continuous, decreasing, and convex function of $z$. Applying the first-order condition for $\hat{\delta}$ leads to

$$\hat{\delta} = \Big( -\varphi'_{\hat{\delta}}(z) + \frac{k}{\beta_1}\Big)^{-\gamma} \frac{\delta K}{z}. \tag{A-16}$$

When the above first-order condition is substituted in (A-15), the differential equation is rewritten as

$$\frac{1}{2}||\beta_3||^2 z^2 \varphi''_{\hat{\delta}}(z) - (\beta_1 - \delta - \beta_2) z \varphi'_{\hat{\delta}}(z) - (\beta_2 + \delta)\varphi_{\hat{\delta}}(z) + \frac{\gamma}{1-\gamma} z^{1-1/\gamma} + z + \frac{\delta K}{1-\gamma}\Big( -\varphi'_{\hat{\delta}}(z) + \frac{k}{\beta_1}\Big)^{1-\gamma} = 0, \quad 0 < z < \overline{z}. \tag{A-17}$$

From now on, we will carry out several transformations to simplify the differential equation given in (A-17). We denote $-\varphi'_{\hat{\delta}}(z)$ by $\tilde{G}(z)$. By differentiating the both sides of (A-17) with respect to $z$, the differential equation (A-17) is restated with $\tilde{G}(z)$ as follows:

$$-\frac{1}{2}||\beta_3||^2 z^2 \tilde{G}''(z) - (||\beta_3||^2 + \beta_2 + \delta - \beta_1) z \tilde{G}'(z) + \beta_1 \tilde{G}(z) + 1 + \delta K \Big(\tilde{G}(z) + \frac{k}{\beta_1}\Big)^{-\gamma} \tilde{G}'(z) = z^{-1/\gamma}, \quad 0 < z < \overline{z}, \tag{A-18}$$

with the boundary conditions

$$\tilde{G}(\overline{z}) = -\frac{L}{\xi I} \text{ and } \tilde{G}'(\overline{z}) = 0.$$

We also denote $\tilde{G}(z) + 1/\beta_1$ by $G(z)$. Then the differential equation (A-18) is rewritten as

$$-\frac{1}{2}||\beta_3||^2 z^2 G''(z) - (||\beta_3||^2 + \beta_2 + \delta - \beta_1) z G'(z) \\ + \beta_1 G(z) + \delta K \Big( G(z) - \frac{1}{\beta_1} + \frac{k}{\beta_1} \Big)^{-\gamma} G'(z) = z^{-1/\gamma}, \quad 0 < z < \overline{z},$$

with the boundary conditions

$$G(\overline{z}) = -\frac{L}{\xi I} + \frac{1}{\beta_1} \quad \text{and} \quad G'(\overline{z}) = 0.$$

Finally, $\hat{\delta}$ given in (A-16) is rewritten as a function of $G(z)$, which completes the proof. **Q.E.D.**

**Remark.** If we solve the equation (A-14), the original value function (8) can be recovered by the relation (A-7) given in Lemma D.4 in which the unique risk-neutral intensity $\hat{\delta}$ is determined by Lemma D.5. The following lemma verifies that the inverse is also true, i.e., the original value function (8) can be converted by the principal of dynamic programming into the solution of the static problem (A-6) that is obtained by solving the equation (A-14).

**Lemma D.6** *The original value function (8) can be converted into the solution of the static problem (A-6).*

**Proof.** By the principal of dynamic programming, the Hamilton-Jacobi-Bellman (HJB) equation associated with the original value function (8) is given by: for any $w > -L \geq -\xi I/\beta_1$,

$$-(\beta + \delta)V + \{rw - c + \xi I\}V_w + \pi^\top(\mu - r\mathbf{1})V_w + \frac{1}{2}||\sigma\pi||^2 V_{ww} \\ + \mu^I I V_I + \frac{1}{2}||\sigma^I||^2 I^2 V_{II} + \pi^\top \sigma^\top \sigma^I I V_{wI} + \frac{c^{1-\gamma}}{1-\gamma} + \delta K \frac{(w + k\xi I/\beta_1)^{1-\gamma}}{1-\gamma} = 0,$$

where $||\cdot||^2 = (\cdot^\top)\cdot$, the subscripts of $V$ represent its partial derivatives. The first-order conditions for consumption $c$ and investment $\pi$ are given by

$$c = V_w^{-1/\gamma}$$

and

$$\pi = -\sigma^{-1}\Big(\theta V_w + \sigma^I I V_{wI}\Big) \Big/ V_{ww},$$

respectively. Substituting the first-order conditions in the above HJB equation, we obtain the following equation: for any $w > -L \geq -\xi I/\beta_1$,

$$
\begin{aligned}
&-(\beta+\delta)V + (rw+\xi I)V_w + \mu^I I V_I + \frac{1}{2}||\sigma^I||^2 I^2 V_{II} + \frac{\gamma}{1-\gamma}V_w^{1-1/\gamma} \\
&-\frac{1}{2}||\theta V_w + \sigma^I I V_{wI}||^2 V_{ww} + \delta K \frac{(w+k\xi I/\beta_1)^{1-\gamma}}{1-\gamma} = 0.
\end{aligned}
$$

Notice that $V(w,I)$ is homogeneous of degree $1-\gamma$ and hence,

$$
V(w,I) = (\xi I)^{1-\gamma} v(z), \quad z = \frac{w}{\xi I}, \tag{A-19}
$$

where $v(z)$ is to be determined. The following relations hold:

$$
\begin{aligned}
&V_w = (\xi I)^{-\gamma} v'(z), \quad V_{ww} = (\xi I)^{-\gamma-1} v''(z), \\
&V_I = (1-\gamma)(\xi I)^{-\gamma} v(z) - (\xi I)^{-\gamma} z v'(z), \\
&V_{II} = -\gamma(1-\gamma)(\xi I)^{-\gamma-1} v(z) + 2\gamma(\xi I)^{-\gamma-1} z v'(z) + (\xi I)^{-\gamma-1} z^2 v''(z), \\
&V_{wI} = -\gamma(\xi I)^{-\gamma-1} v'(z) - (\xi I)^{-\gamma-1} z v''(z).
\end{aligned}
$$

As a result, we now obtain the following one-dimensional HJB equation: for any $z > -L/(\xi I)$,

$$
\beta_1 z v'(z) - \beta_2 v(z) - \frac{\gamma}{1-\gamma} v'(z)^{1-1/\gamma} + \delta K \frac{(z+k/\beta_1)^{1-\gamma}}{1-\gamma} - \frac{1}{2}||\beta_3||^2 \frac{v'(z)^2}{v''(z)} = 0, \tag{A-20}
$$

where

$$
\begin{aligned}
&\beta_1 = r - \mu^I + (\sigma^I)^\top \theta, \\
&\beta_2 = \beta - (1-\gamma)\mu^I + \frac{1}{2}\gamma(1-\gamma)||\sigma^I||^2, \\
&\beta_3 = \gamma(\sigma^I)^\top - \theta^\top.
\end{aligned}
$$

We now utilize the convex-duality approach of Bensoussan *et al.* (2016). We first introduce the dual variable $\lambda(z)$ defined as the first derivative of $v(z)$:

$$
\lambda(z) \equiv v'(z), \tag{A-21}
$$

which serves as the agent's marginal value function. We then introduce the convex-dual function $G(\lambda(z))$ whose multiplication by income $I$ is defined as total wealth that is the sum of financial wealth $w$ and the present value $\xi I/\beta_1$ of future income $I$, which is the certainty equivalent present value of lifetime labor income:

$$
\xi I G(\lambda(z)) \equiv w + \frac{\xi I}{\beta_1}
$$

or equivalently,

$$G(\lambda(z)) = z + \frac{1}{\beta_1}. \tag{A-22}$$

The convex-dual function $G(\lambda(z))$ satisfies the following relations:

$$G'(\lambda(z))\lambda'(z) = 1, \quad G''(\lambda(z))\lambda'(z)^2 + G'(\lambda(z))\lambda''(z) = 0. \tag{A-23}$$

For notational simplicity, we will write $G(\lambda(z))$ as $G(\lambda)$ and $\lambda(z)$ as $\lambda$ if there is no any confusion. After taking a differentiation on the both sides of the HJB equation (A-20) with respect to $z$, we can obtain the following dual HJB equation with the convex-dual function $G(\lambda)$ and the dual variable $\lambda$: $0 < \lambda < \overline{\lambda}$,

$$\begin{aligned} -\frac{1}{2}||\beta_3||^2\lambda^2 G''(\lambda) - (||\beta_3||^2 + \beta_2 + \delta - \beta_1)\lambda G'(\lambda) \\ + \beta_1 G(\lambda) + \delta K\Big(G(\lambda) - \frac{1}{\beta_1} + \frac{k}{\beta_1}\Big)^{-\gamma} G'(\lambda) = \lambda^{-1/\gamma}, \end{aligned} \tag{A-24}$$

where $\overline{\lambda}$ is a constant to be determined by the value matching and smooth pasting conditions as follows:

$$G(\overline{\lambda}) = -\frac{L}{\xi I} + \frac{1}{\beta_1} \quad \text{and} \quad G'(\overline{\lambda}) = 0.$$

The convex-dual function $G(\lambda)$ obtained from the original value function (8) is, therefore, exactly the same as $G(z)$ solving the equation (A-14) that recovers the solution of the static problem (A-6). We therefore conclude that the original value function (8) can be converted into the solution of the static problem (A-6).

Notice that the original value function (8) is also recovered from the convex-dual function $G(\lambda)$ by the primal HJB equation (A-20) as follows:

$$\begin{aligned} v(z) &= \frac{1}{\beta_2}\Big[\beta_1 z v'(z) - \frac{\gamma}{1-\gamma} v'(z)^{1-1/\gamma} + \delta K \frac{(z + k/\beta_1)^{1-\gamma}}{1-\gamma} - \frac{1}{2}||\beta_3||^2 \frac{v'(z)^2}{v''(z)}\Big] \\ &= \frac{1}{\beta_2}\Big[\beta_1\lambda\Big(G(\lambda) - \frac{1}{\beta_1}\Big) - \frac{\gamma}{1-\gamma}\lambda^{1-1/\gamma} + \delta K \frac{\big(G(\lambda) - 1/\beta_1 + k/\beta_1\big)^{1-\gamma}}{1-\gamma} - \frac{1}{2}||\beta_3||^2\lambda^2 G'(\lambda)\Big], \end{aligned}$$

where the second equality results from (A-21), (A-22), and (A-23), so that the original value function $V(w, I)$ is now obtained by (A-19).

# E Proof of Theorem 3.2

**Proposition E.1** *A general solution of the differential equation (A-14) is given by*

$$\begin{aligned} G(z) =& \frac{1}{\hat{A}+\delta/\gamma} z^{-1/\gamma} + B_\delta^* z^{-\alpha_\delta^*} \\ &+ \frac{2\delta K}{||\beta_3||^2(\alpha_\delta - \alpha_\delta^*)(1-\gamma)} \Big[ (\alpha_\delta - 1) z^{-\alpha_\delta} \int_0^z \mu^{\alpha_\delta - 2} \Big( G(\mu) - \frac{1}{\beta_1} + \frac{k}{\beta_1} \Big)^{1-\gamma} d\mu \\ &+ (\alpha_\delta^* - 1) z^{-\alpha_\delta^*} \int_z^{\overline{z}} \mu^{\alpha_\delta^* - 2} \Big( G(\mu) - \frac{1}{\beta_1} + \frac{k}{\beta_1} \Big)^{1-\gamma} d\mu \Big], \end{aligned} \qquad \text{(A-25)}$$

*where $\hat{A}$ is given by*

$$\hat{A} = \frac{\gamma - 1}{\gamma} \Big( \beta_1 + \frac{||\beta_3||^2}{2\gamma} \Big) + \frac{\beta_2}{\gamma}, \qquad \text{(A-26)}$$

*$B_\delta^*$ and $\overline{z}$ are the two constants to be determined by the boundary conditions:*

$$G(\overline{z}) = -\frac{L}{\xi I} + \frac{1}{\beta_1} \quad \textit{and} \quad G'(\overline{z}) = 0, \qquad \text{(A-27)}$$

*and $\alpha_\delta > 1$ and $-1 < \alpha_\delta^* < 0$ are the two roots to the following characteristic equation:*

$$F(\alpha; \delta) \equiv -\frac{1}{2} ||\beta_3||^2 \alpha(\alpha - 1) + (\beta_2 + \delta - \beta_1)\alpha + \beta_1 = 0. \qquad \text{(A-28)}$$

**Proof.** We conjecture a general solution of the equation (A-14) as

$$G(z) = \frac{1}{\hat{A}+\delta/\gamma} z^{-1/\gamma} + \eta(z) z^{-\alpha_\delta} + \eta^*(z) z^{-\alpha_\delta^*}, \qquad \text{(A-29)}$$

subject to

$$\eta'(z) z^{-\alpha_\delta} + (\eta^*(z))' z^{-\alpha_\delta^*} = 0,$$

where $\alpha_\delta > 1$ and $-1 < \alpha_\delta^* < 0$ are the two roots to the following characteristic equation:

$$F(\alpha; \delta) \equiv -\frac{1}{2} ||\beta_3||^2 \alpha(\alpha - 1) + (\beta_2 + \delta - \beta_1)\alpha + \beta_1 = 0.$$

Direct calculations of the first and second derivative of $G$ result in

$$G'(z) = -\frac{1}{\gamma(\hat{A}+\delta/\gamma)} z^{-1/\gamma - 1} - \alpha_\delta \eta(z) z^{-\alpha_\delta - 1} - \alpha_\delta^* \eta^*(z) z^{-\alpha_\delta^* - 1}$$

and

$$\begin{aligned} G''(z) = \Big(1 + \frac{1}{\gamma}\Big) \frac{1}{\gamma(\hat{A}+\delta/\gamma)} z^{-1/\gamma - 2} - \alpha_\delta \eta'(z) z^{-\alpha_\delta - 1} + \alpha_\delta(\alpha_\delta + 1)\eta(z) z^{-\alpha_\delta - 2} \\ - \alpha_\delta^* (\eta^*(z))' z^{-\alpha_\delta^* - 1} + \alpha_\delta^*(\alpha_\delta^* + 1)\eta^*(z) z^{-\alpha_\delta^* - 2}. \end{aligned}$$

Using the general solution (A-29) and the derivatives of $G$ stated above, the first three terms of left-hand side in (A-14) become

$$
\begin{aligned}
&-\frac{1}{2}||\beta_3||^2 z^2 G''(z) - (||\beta_3||^2 + \beta_2 + \delta - \beta_1) z G'(z) + \beta_1 G(z) \\
&= z^{-1/\gamma} + \frac{||\beta_3||^2}{2}(\alpha_\delta - \alpha_\delta^*) z^{1-\alpha_\delta} \eta'(z) \\
&= z^{-1/\gamma} - \frac{||\beta_3||^2}{2}(\alpha_\delta - \alpha_\delta^*) z^{1-\alpha_\delta^*} (\eta^*(z))'.
\end{aligned}
$$

As a result, the differential equation (A-14) simplifies to the following: for $0 < z < \overline{z}$,

$$
\frac{||\beta_3||^2}{2}(\alpha_\delta - \alpha_\delta^*) z^{1-\alpha_\delta} \eta'(z) = -\delta K \Big( G(z) - \frac{1}{\beta_1} + \frac{k}{\beta_1} \Big)^{-\gamma} G'(z)
$$

and

$$
\frac{||\beta_3||^2}{2}(\alpha_\delta - \alpha_\delta^*) z^{1-\alpha_\delta^*} (\eta^*(z))' = \delta K \Big( G(z) - \frac{1}{\beta_1} + \frac{k}{\beta_1} \Big)^{-\gamma} G'(z).
$$

Integrating the both sides of the above two relationships from 0 to $z$ and from $z$ to $\overline{z}$ allows $\eta(z)$ and $\eta^*(z)$ to be expressed as an integral form:

$$
\eta(z) = -\frac{2\delta K}{||\beta_3||^2(\alpha_\delta - \alpha_\delta^*)} \int_0^z \mu^{\alpha_\delta - 1} \Big( G(\mu) - \frac{1}{\beta_1} + \frac{k}{\beta_1} \Big)^{-\gamma} G'(\mu) d\mu
$$

and

$$
\eta^*(z) = \eta^*(\overline{z}) - \frac{2\delta K}{||\beta_3||^2(\alpha_\delta - \alpha_\delta^*)} \int_z^{\overline{z}} \mu^{\alpha_\delta^* - 1} \Big( G(\mu) - \frac{1}{\beta_1} + \frac{k}{\beta_1} \Big)^{-\gamma} G'(\mu) d\mu.
$$

Therefore, the general solution (A-29) also can be expressed as an integral form:

$$
\begin{aligned}
G(z) = &\frac{1}{\hat{A} + \delta/\gamma} z^{-1/\gamma} + \eta^*(\overline{z}) z^{-\alpha_\delta^*} - \frac{2\delta K}{||\beta_3||^2(\alpha_\delta - \alpha_\delta^*)} \Big[ z^{-\alpha_\delta} \int_0^z \mu^{\alpha_\delta - 1} \Big( G(\mu) - \frac{1}{\beta_1} + \frac{k}{\beta_1} \Big)^{-\gamma} G'(\mu) d\mu \\
&+ z^{-\alpha_\delta^*} \int_z^{\overline{z}} \mu^{\alpha_\delta^* - 1} \Big( G(\mu) - \frac{1}{\beta_1} + \frac{k}{\beta_1} \Big)^{-\gamma} G'(\mu) d\mu \Big].
\end{aligned}
\tag{A-30}
$$

Note that

$$
\Big( G(\mu) - \frac{1}{\beta_1} + \frac{k}{\beta_1} \Big)^{-\gamma} G'(\mu) = \frac{d}{d\mu} \Big\{ \frac{1}{1-\gamma} \Big( G(\mu) - \frac{1}{\beta_1} + \frac{k}{\beta_1} \Big)^{1-\gamma} \Big\}.
$$

Using the integration by parts, the general solution (A-30) can be restated as follows:

$$
\begin{aligned}
G(z) = &\frac{1}{\hat{A} + \delta/\gamma} z^{-1/\gamma} + \Big\{ \eta^*(\overline{z}) + \overline{z}^{\alpha_\delta^* - 1} \frac{1}{1-\gamma} \Big( G(\overline{z}) - \frac{1}{\beta_1} + \frac{k}{\beta_1} \Big)^{1-\gamma} \Big\} z^{-\alpha_\delta^*} \\
&+ \frac{2\delta K}{||\beta_3||^2(\alpha_\delta - \alpha_\delta^*)(1-\gamma)} \Big[ (\alpha_\delta - 1) z^{-\alpha_\delta} \int_0^z \mu^{\alpha_\delta - 2} \Big( G(\mu) - \frac{1}{\beta_1} + \frac{k}{\beta_1} \Big)^{1-\gamma} d\mu \\
&+ (\alpha_\delta^* - 1) z^{-\alpha_\delta^*} \int_z^{\overline{z}} \mu^{\alpha_\delta^* - 2} \Big( G(\mu) - \frac{1}{\beta_1} + \frac{k}{\beta_1} \Big)^{1-\gamma} d\mu \Big].
\end{aligned}
$$

Defining a constant $B_\delta^*$ as

$$B_\delta^* \equiv \eta^*(\overline{z}) + \overline{z}^{\alpha_\delta^*-1}\frac{1}{1-\gamma}\Big(G(\overline{z}) - \frac{1}{\beta_1} + \frac{k}{\beta_1}\Big)^{1-\gamma}.$$

Finally, we obtain the general solution in closed-form:

$$\begin{aligned} G(z) =& \frac{1}{\hat{A}+\delta/\gamma} z^{-1/\gamma} + B_\delta^* z^{-\alpha_\delta^*} \\ &+ \frac{2\delta K}{||\beta_3||^2(\alpha_\delta - \alpha_\delta^*)(1-\gamma)}\Big[(\alpha_\delta - 1) z^{-\alpha_\delta}\int_0^z \mu^{\alpha_\delta - 2}\Big(G(\mu) - \frac{1}{\beta_1} + \frac{k}{\beta_1}\Big)^{1-\gamma} d\mu \\ &+ (\alpha_\delta^* - 1) z^{-\alpha_\delta^*}\int_z^{\overline{z}} \mu^{\alpha_\delta^* - 2}\Big(G(\mu) - \frac{1}{\beta_1} + \frac{k}{\beta_1}\Big)^{1-\gamma} d\mu\Big], \end{aligned}$$

which completes the proof. **Q.E.D.**

With the first-order condition (A-13) for $\hat{\delta}$, the first-order conditions for consumption $c(t)$ in (A-11) can be rewritten as

$$c(t) = \xi I(t)\Gamma^{\hat{\delta}}(t)^{-1/\gamma}, \tag{A-31}$$

where $\Gamma^{\hat{\delta}}(t)$ is given by

$$\Gamma^{\hat{\delta}}(t) = \lambda e^{(\beta+\delta-\hat{\delta})t} H(t)(\xi I(t))^\gamma,$$

and

$$\hat{\delta} = \Big(G(\Gamma^{\hat{\delta}}(t)) - \frac{1}{\beta_1} + \frac{k}{\beta_1}\Big)^{-\gamma}\frac{\delta K}{\Gamma^{\hat{\delta}}(t)}.$$

By the principle of dynamic programming, it is convenient to express the consumption as a function of initial variable $z$:

$$c(t) = c(0) = \xi I z^{-1/\gamma}. \tag{A-32}$$

From the relationship (A-7) between the value function and the indirect value function, applying the first-order condition for $\lambda$ results in

$$w = -J_\lambda(\lambda, I) = -\xi I \varphi'_{\hat{\delta}}(z) = \xi I \tilde{G}(z) = \xi I\Big(G(z) - \frac{1}{\beta_1}\Big), \tag{A-33}$$

accordingly,

$$G(z) = \frac{w}{\xi I} + \frac{1}{\beta_1}.$$

A little rearrangement of the general solution (A-25) leads to

$$
\begin{aligned}
z^{-1/\gamma} &= (\hat{A}+\delta/\gamma)\Big[G(z) - B_\delta^* z^{-\alpha_\delta^*} \\
&\qquad - \frac{2\delta K}{||\beta_3||^2(\alpha_\delta-\alpha_\delta^*)(1-\gamma)}\Big[(\alpha_\delta-1)z^{-\alpha_\delta}\int_0^z \mu^{\alpha_\delta-2}\Big(G(\mu)-\frac{1}{\beta_1}+\frac{k}{\beta_1}\Big)^{1-\gamma}d\mu \\
&\qquad + (\alpha_\delta^*-1)z^{-\alpha_\delta^*}\int_z^{\overline{z}} \mu^{\alpha_\delta^*-2}\Big(G(\mu)-\frac{1}{\beta_1}+\frac{k}{\beta_1}\Big)^{1-\gamma}d\mu\Big] \\
&= (\hat{A}+\delta/\gamma)\Big[\frac{w}{\xi I}+\frac{1}{\beta_1} - B_\delta^* z^{-\alpha_\delta^*} \\
&\qquad - \frac{2\delta K}{||\beta_3||^2(\alpha_\delta-\alpha_\delta^*)(1-\gamma)}\Big[(\alpha_\delta-1)z^{-\alpha_\delta}\int_0^z \mu^{\alpha_\delta-2}\Big(G(\mu)-\frac{1}{\beta_1}+\frac{k}{\beta_1}\Big)^{1-\gamma}d\mu \\
&\qquad + (\alpha_\delta^*-1)z^{-\alpha_\delta^*}\int_z^{\overline{z}} \mu^{\alpha_\delta^*-2}\Big(G(\mu)-\frac{1}{\beta_1}+\frac{k}{\beta_1}\Big)^{1-\gamma}d\mu\Big].
\end{aligned}
$$

Therefore, the first-order condition for consumption $c(t)$ in (A-31) allows the following optimal consumption strategy:

$$c(t) = (\hat{A}+\delta/\gamma)\Big(w+\frac{\xi I}{\beta_1} - \xi I B_\delta^* z^{-\alpha_\delta^*} - \mathrm{IP}\Big), \tag{A-34}$$

where IP represents the integral parts given by

$$\mathrm{IP} = \mathrm{IP1}+\mathrm{IP2}, \tag{A-35}$$

$$
\begin{aligned}
\mathrm{IP1} &= \frac{2\delta K(\alpha_\delta-1)\xi I}{||\beta_3||^2(\alpha_\delta-\alpha_\delta^*)(1-\gamma)} z^{-\alpha_\delta}\int_0^z \mu^{\alpha_\delta-2}\Big(G(\mu)-\frac{1}{\beta_1}+\frac{k}{\beta_1}\Big)^{1-\gamma}d\mu < 0, \\
\mathrm{IP2} &= \frac{2\delta K(\alpha_\delta^*-1)\xi I}{||\beta_3||^2(\alpha_\delta-\alpha_\delta^*)(1-\gamma)} z^{-\alpha_\delta^*}\int_z^{\overline{z}} \mu^{\alpha_\delta^*-2}\Big(G(\mu)-\frac{1}{\beta_1}+\frac{k}{\beta_1}\Big)^{1-\gamma}d\mu > 0.
\end{aligned}
$$

It remains to derive the optimal investment strategy. A little rearrangement of the relationship in (A-33) gives

$$\frac{w}{\xi I} = G(z) - \frac{1}{\beta_1},$$

or equivalently,

$$\frac{W(t)}{\xi I(t)} = G(\Gamma^{\hat{\delta}}(t)) - \frac{1}{\beta_1}. \tag{A-36}$$

By applying It$\hat{o}$'s formula to the left hand side of the above relationship,

$$
\begin{aligned}
d\Big(\frac{W(t)}{\xi I(t)}\Big) &= dW(t)\frac{1}{\xi I(t)} + W(t)d\Big(\frac{1}{\xi I(t)}\Big) + dW(t)d\Big(\frac{1}{\xi I(t)}\Big) \\
&= \Big[\Big\{rW(t) - c(t) + \xi I(t) + \pi(t)^\top(\mu - r\mathbf{1})\Big\}dt + \pi(t)^\top\sigma^\top dZ(t)\Big]\frac{1}{\xi I(t)} \\
&\quad + W(t)\Big[-(\xi I(t))^{-2}dI(t) + (\xi I(t))^{-3}(dI(t))^2\Big] \\
&\quad + \Big[\Big\{rW(t) - c(t) + \xi I(t) + \pi(t)^\top(\mu - r\mathbf{1})\Big\}dt + \pi(t)^\top\sigma^\top dZ(t)\Big] \\
&\quad \times \Big[-(\xi I(t))^{-2}dI(t) + (\xi I(t))^{-3}(dI(t))^2\Big] \\
&= \Big[\Big\{rW(t) - c(t) + \xi I(t) + \pi(t)^\top(\mu - r\mathbf{1})\Big\}dt + \pi(t)^\top\sigma^\top dZ(t)\Big]\frac{1}{\xi I(t)} \\
&\quad + W(t)\Big[-(\xi I(t))^{-1}\{\mu^I dt + (\sigma^I)^\top dZ(t)\} + (\xi I(t))^{-1}||\sigma^I||^2 dt\Big] \\
&\quad + \Big[\Big\{rW(t) - c(t) + \xi I(t) + \pi(t)^\top(\mu - r\mathbf{1})\Big\}dt + \pi(t)^\top\sigma^\top dZ(t)\Big] \\
&\quad \times \Big[-(\xi I(t))^{-1}\{\mu^I dt + (\sigma^I)^\top dZ(t)\} + (\xi I(t))^{-1}||\sigma^I||^2 dt\Big] \\
&= \Big[\Big\{rW(t) - c(t) + \xi I(t) + \pi(t)^\top(\mu - r\mathbf{1})\Big\}dt + \pi(t)^\top\sigma^\top dZ(t)\Big]\frac{1}{\xi I(t)} \\
&\quad + \frac{W(t)}{\xi I(t)}\Big[-(\mu^I - ||\sigma^I||^2)dt - (\sigma^I)^\top dZ(t)\Big] - (\xi I(t))^{-1}\pi(t)^\top\sigma^\top\sigma^I dt \\
&= \frac{1}{\xi I(t)}\Big[\Big\{rW(t) - c(t) + \xi I(t) + \pi(t)^\top(\mu - r\mathbf{1}) - W(t)(\mu^I - ||\sigma^I||^2) - \pi(t)^\top\sigma^\top\sigma^I\Big\}dt \\
&\qquad + \{\pi(t)^\top\sigma^\top - W(t)(\sigma^I)^\top\}dZ(t)\Big] \\
&= \frac{1}{\xi I(t)}\Big[\Big\{\{r - \mu^I + ||\sigma^I||^2\}W(t) - c(t) + \xi I(t) + \pi(t)^\top\{\mu - r\mathbf{1} - \sigma^\top\sigma^I\}\Big\}dt \\
&\qquad + \{\pi(t)^\top\sigma^\top - W(t)(\sigma^I)^\top\}dZ(t)\Big].
\end{aligned}
$$

By applying It$\hat{o}$'s formula to the right hand side of the relationship (A-36),

$$
\begin{aligned}
dG(\Gamma^{\hat{\delta}}(t)) &= G'(\Gamma^{\hat{\delta}}(t))d\Gamma^{\hat{\delta}}(t) + \frac{1}{2}G''(\Gamma^{\hat{\delta}}(t))(d\Gamma^{\hat{\delta}}(t))^2 \\
&= G'(\Gamma^{\hat{\delta}}(t))\Gamma^{\hat{\delta}}(t)\{-(\beta_1^{\hat{\delta}} - \beta_2)dt + \beta_3 d\tilde{Z}(t)\} + \frac{1}{2}G''(\Gamma^{\hat{\delta}}(t))(\Gamma^{\hat{\delta}}(t))^2||\beta_3||^2 dt \\
&= G'(\Gamma^{\hat{\delta}}(t))\Gamma^{\hat{\delta}}(t)\{-(\beta_1^{\hat{\delta}} - \beta_2)dt + \beta_3\{-(1-\gamma)\sigma^I dt + dZ(t)\} + \frac{1}{2}G''(\Gamma^{\hat{\delta}}(t))(\Gamma^{\hat{\delta}}(t))^2||\beta_3||^2 dt \\
&= \Big\{-G'(\Gamma^{\hat{\delta}}(t))\Gamma^{\hat{\delta}}(t)\{(\beta_1^{\hat{\delta}} - \beta_2) + (1-\gamma)\beta_3\sigma^I\} + \frac{1}{2}G''(\Gamma^{\hat{\delta}}(t))(\Gamma^{\hat{\delta}}(t))^2||\beta_3||^2\Big\}dt \\
&\quad + G'(\Gamma^{\hat{\delta}}(t))\Gamma^{\hat{\delta}}(t)\beta_3 dZ(t).
\end{aligned}
\tag{A-37}
$$

Equating each term of $dZ(t)$ in $d(W(t)/(\xi I(t)))$ and $dG(\Gamma^{\hat{\delta}}(t))$ derives the following relationship that involves the optimal investment strategy $\pi(t)$:

$$\frac{\pi(t)^\top\sigma^\top - W(t)(\sigma^I)^\top}{\xi I(t)} = G'(\Gamma^{\hat{\delta}}(t))\Gamma^{\hat{\delta}}(t)\beta_3. \tag{A-38}$$

By the principle of dynamic programming, it is convenient to express the investment as a function of initial variables at time 0:

$$\frac{\pi^\top\sigma^\top - w(\sigma^I)^\top}{\xi I} = G'(z)z\beta_3, \tag{A-39}$$

where $\pi = \pi(t) = \pi(0)$. Using the general solution $G(z)$ given in (A-25), a direct calculation of $G'(z)$ yields

$$\begin{aligned}G'(z) =& -\frac{1}{\gamma(\hat{A}+\delta/\gamma)}z^{-1/\gamma-1} - \alpha_\delta^* B_\delta^* z^{-\alpha_\delta^*-1} + \frac{2\delta K}{||\beta_3||^2(1-\gamma)z^2}\Big(G(z) - \frac{1}{\beta_1} + \frac{k}{\beta_1}\Big)^{1-\gamma}\\ &- \frac{2\delta K\alpha_\delta(\alpha_\delta-1)}{||\beta_3||^2(\alpha_\delta-\alpha_\delta^*)(1-\gamma)}z^{-\alpha_\delta-1}\int_0^z \mu^{\alpha_\delta-2}\Big(G(\mu) - \frac{1}{\beta_1} + \frac{k}{\beta_1}\Big)^{1-\gamma}d\mu\\ &- \frac{2\delta K\alpha_\delta^*(\alpha_\delta^*-1)}{||\beta_3||^2(\alpha_\delta-\alpha_\delta^*)(1-\gamma)}z^{-\alpha_\delta^*-1}\int_z^{\bar{z}} \mu^{\alpha_\delta^*-2}\Big(G(\mu) - \frac{1}{\beta_1} + \frac{k}{\beta_1}\Big)^{1-\gamma}d\mu.\end{aligned}$$

Multiplying $G'(z)$ by $\xi Iz$ gives

$$\begin{aligned}\xi IG'(z)z =& -\frac{1}{\gamma(\hat{A}+\delta/\gamma)}\xi Iz^{-1/\gamma} - \alpha_\delta^*\xi IB_\delta^* z^{-\alpha_\delta^*}\\ &+ \frac{2\delta K\xi I}{||\beta_3||^2(1-\gamma)z}\Big(G(z) - \frac{1}{\beta_1} + \frac{k}{\beta_1}\Big)^{1-\gamma}\\ &- \alpha_\delta \times \text{IP1} - \alpha_\delta^* \times \text{IP2},\end{aligned}$$

where IP1 and IP2 given in (A-35) are the first and second integral part of income-disaster-induced precautionary savings. Note that $\xi Iz^{-1/\gamma}$ of the first term in the above relationship is equivalent to the optimal consumption strategy from (A-32), as a result, $\xi IG'(z)z$ can be restated with (A-34) as the following:

$$\begin{aligned}\xi IG'(z)z = -\frac{1}{\gamma}\Big[&w + \frac{\xi I}{\beta_1} + (\gamma\alpha_\delta^*-1)\xi IB_\delta^* z^{-\alpha_\delta^*}\\ &- \frac{2\gamma}{||\beta_3||^2(1-\gamma)z}\xi I\delta K\Big(G(z) - \frac{1}{\beta_1} + \frac{k}{\beta_1}\Big)^{1-\gamma}\\ &+ (\gamma\alpha_\delta - 1)\times \text{IP1} + (\gamma\alpha_\delta^*-1)\times\text{IP2}\Big].\end{aligned}$$

Note that

$$
\begin{aligned}
&-\frac{2\gamma}{||\beta_3||^2(1-\gamma)z}\xi I\delta K\Big(G(z)-\frac{1}{\beta_1}+\frac{k}{\beta_1}\Big)^{1-\gamma}\\
&=-\frac{2\gamma}{||\beta_3||^2 z}\xi I\delta K\frac{\Big(\frac{w}{\xi I}+\frac{k}{\beta_1}\Big)^{1-\gamma}}{1-\gamma}\\
&=-\frac{2\gamma}{||\beta_3||^2 z}(\xi I)^{\gamma}\delta K\frac{\Big(w+\frac{k\xi I}{\beta_1}\Big)^{1-\gamma}}{1-\gamma}\\
&=-\frac{2\gamma}{||\beta_3||^2}\delta K\frac{\Big(w+\frac{k\xi I}{\beta_1}\Big)^{1-\gamma}}{1-\gamma}c(t)^{\gamma}\\
&=-\frac{2\gamma}{||\beta_3||^2}\delta K\frac{\Big(w+\frac{k\xi I}{\beta_1}\Big)^{1-\gamma}}{1-\gamma}\Big/c(t)^{-\gamma}.
\end{aligned}
$$

Therefore, we derive the optimal investment strategy from (A-39):

$$
\begin{aligned}
\pi(t)&=\sigma^{-1}(\beta_3)^{\top}\xi IG'(z)z+\sigma^{-1}\sigma^I w\\
&=\sigma^{-1}(\gamma\sigma^I-\theta)\xi IG'(z)z+\sigma^{-1}\sigma^I w\\
&=\frac{1}{\gamma}\sigma^{-1}\theta w\\
&\quad+\frac{1}{\gamma}\sigma^{-1}(\theta-\gamma\sigma^I)\Big[\frac{\xi I}{\beta_1}+(\gamma\alpha_\delta^*-1)\xi IB_\delta^* z^{-\alpha_\delta^*}\\
&\quad-\frac{2\gamma}{||\beta_3||^2}\delta K\frac{\Big(w+\frac{k\xi I}{\beta_1}\Big)^{1-\gamma}}{1-\gamma}\Big/c(t)^{-\gamma}\\
&\quad+(\gamma\alpha_\delta-1)\times \mathrm{IP1}+(\gamma\alpha_\delta^*-1)\times \mathrm{IP2}\Big].
\end{aligned}
$$

Following Karatzas *et al.* (1991), the optimality would be verified if the wealth process $W(t)$

was self financed by $c(t)$ and $\pi(t)$. The term of $dt$ of $dG(\Gamma^{\hat{\delta}}(t))$ in (A-37) is rewritten as

$$
\begin{aligned}
&- G'(\Gamma^{\hat{\delta}}(t))\Gamma^{\hat{\delta}}(t)\{(\beta_1^{\hat{\delta}} - \beta_2) + (1-\gamma)\beta_3\sigma^I\} + \frac{1}{2}G''(\Gamma^{\hat{\delta}}(t))(\Gamma^{\hat{\delta}}(t))^2||\beta_3||^2 \\
&= \frac{1}{2}||\beta_3||^2(\Gamma^{\hat{\delta}}(t))^2G''(\Gamma^{\hat{\delta}}(t)) + \{\beta_2 - \beta_1 - \hat{\delta} + \delta - (1-\gamma)\beta_3\sigma^I\}\Gamma^{\hat{\delta}}(t)G'(\Gamma^{\hat{\delta}}(t)) \\
&= \frac{1}{2}||\beta_3||^2(\Gamma^{\hat{\delta}}(t))^2G''(\Gamma^{\hat{\delta}}(t)) + (\beta_2 + \delta - \beta_1)\Gamma^{\hat{\delta}}(t)G'(\Gamma^{\hat{\delta}}(t)) - (1-\gamma)\beta_3\sigma^I\Gamma^{\hat{\delta}}(t)G'(\Gamma^{\hat{\delta}}(t)) \\
&\quad - \Big(G(\Gamma^{\hat{\delta}}(t)) - \frac{1}{\beta_1} + \frac{k}{\beta_1}\Big)^{-\gamma}\frac{\delta K}{\Gamma^{\hat{\delta}}(t)}\Gamma^{\hat{\delta}}(t)G'(\Gamma^{\hat{\delta}}(t)) \\
&= \frac{1}{2}||\beta_3||^2(\Gamma^{\hat{\delta}}(t))^2G''(\Gamma^{\hat{\delta}}(t)) + (\beta_2 + \delta - \beta_1)\Gamma^{\hat{\delta}}(t)G'(\Gamma^{\hat{\delta}}(t)) - \delta K\Big(G(\Gamma^{\hat{\delta}}(t)) - \frac{1}{\beta_1} + \frac{k}{\beta_1}\Big)^{-\gamma}G'(\Gamma^{\hat{\delta}}(t)) \\
&\quad - (1-\gamma)\Gamma^{\hat{\delta}}(t)G'(\Gamma^{\hat{\delta}}(t))\beta_3\sigma^I \\
&= -||\beta_3||^2\Gamma^{\hat{\delta}}(t)G'(\Gamma^{\hat{\delta}}(t)) + \beta_1G(\Gamma^{\hat{\delta}}(t)) - \Gamma^{\hat{\delta}}(t)^{-1/\gamma} - (1-\gamma)\Gamma^{\hat{\delta}}(t)G'(\Gamma^{\hat{\delta}}(t))\beta_3\sigma^I \\
&= -\frac{\pi(t)^\top\sigma^\top - W(t)(\sigma^I)^\top}{\xi I(t)}(\beta_3)^\top + \beta_1\Big(\frac{W(t)}{\xi I(t)} + \frac{1}{\beta_1}\Big) - \frac{c(t)}{\xi I(t)} - (1-\gamma)\frac{\pi(t)^\top\sigma^\top - W(t)(\sigma^I)^\top}{\xi I(t)}\sigma^I \\
&= \frac{1}{\xi I(t)}\Big[\{(\sigma^I)^\top(\beta_3)^\top + \beta_1\}W(t) - c(t) + \xi I(t) - \pi(t)^\top\sigma^\top(\beta_3)^\top - (1-\gamma)\pi(t)^\top\sigma^\top\sigma^I + (1-\gamma)||\sigma^I||^2W(t)\Big] \\
&= \frac{1}{\xi I(t)}\Big[\{(\sigma^I)^\top(\gamma\sigma^I - \theta) + r - \mu^I + (\sigma^I)^\top\theta\}W(t) - c(t) + \xi I(t) - \pi(t)^\top\sigma^\top(\gamma\sigma^I - \theta) \\
&\qquad - (1-\gamma)\pi(t)^\top\sigma^\top\sigma^I + (1-\gamma)||\sigma^I||^2W(t)\Big] \\
&= \frac{1}{\xi I(t)}\Big[\{r - \mu^I + \gamma||\sigma^I||^2\}W(t) - c(t) + \xi I(t) + \pi(t)^\top\{\mu - r\mathbf{1} - \sigma^\top\sigma^I\}\Big].
\end{aligned}
$$

where the second equality derives when $\hat{\delta}$ in (A-13) substituted in, the fourth equality derives from the differential equation in (A-14), the fifth equality derives from $||\beta_3||^2 = \beta(\beta_3)^\top$, (A-31), (A-36), and (A-38). This shows that each term of $dt$ in $d(W(t)/(\xi I(t)))$ and $dG(\Gamma^{\hat{\delta}}(t))$ are exactly the same, as a result, the wealth process $W(t)$ is self financed by the optimal consumption strategy $c(t)$ and the optimal investment strategy $\pi(t)$ with the risk-neutral intensity $\hat{\delta}$ in (A-13).

# F Proof of Theorem 4.1

The optimal consumption strategies for the normal agent ($n$) and the income-disaster-exposed agent ($d$) prior to income disaster derive from (A-11): $t < \tau$,

$$
\begin{aligned}
c_n(t) &= \Big(\lambda_n e^{(\beta+\delta-\hat{\delta})t}H(t)\Big)^{-1/\gamma}, \\
c_d(t) &= \Big(\lambda_d e^{(\beta+\delta-\hat{\delta})t}H(t)\Big)^{-1/\gamma},
\end{aligned}
\tag{A-40}
$$

where the constants $\lambda_n$ and $\lambda_d$ should satisfy

$$E\Big[\int_0^\infty e^{-\hat{\delta}t}H(t)\Big(c_n(t) - \xi_n I_n(t) + \hat{\delta}W_n(t)\Big)dt\Big] = w_n,$$
$$E\Big[\int_0^\infty e^{-\hat{\delta}t}H(t)\Big(c_d(t) - \xi_d I(t) + \hat{\delta}W_d(t)\Big)dt\Big] = w_d,$$

where $\xi = \xi_n + \xi_d$, $w = w_n + w_d$, with the following optimal wealth processes for agent $n$ and agent $d$ prior to income disaster: $t < \tau$,

$$W_n(t) = \Big(\lambda_d e^{(\beta+\delta-\hat{\delta})t}H(t)\Big)^{-1/\gamma}\Big(\frac{\hat{\delta}}{\delta}\Big)^{-1/\gamma}K^{1/\gamma} - \frac{k\xi_n I_n(t)}{\beta_1},$$
$$W_d(t) = \Big(\lambda_d e^{(\beta+\delta-\hat{\delta})t}H(t)\Big)^{-1/\gamma}\Big(\frac{\hat{\delta}}{\delta}\Big)^{-1/\gamma}K^{1/\gamma} - \frac{k\xi_d I(t)}{\beta_1}.$$

According to the clearing condition of consumption good given in Definition 4.1, $c_n(t)+c_d(t) = I(t)$, the equilibrium SDF $H(t)$ prior to income disaster follows: $t < \tau$,

$$H(t) = \{(\lambda_n)^{-1/\gamma} + (\lambda_d)^{-1/\gamma}\}^\gamma e^{-(\beta-(\hat{\delta}-\delta))t}I(t)^{-\gamma}, \tag{A-41}$$

which completes the proof. **Q.E.D.**

# G Proof of Theorem 4.2

The proof is straightforward by direct calculation. **Q.E.D.**

# H Proof of Theorem 4.3

The equilibrium stock return dynamics before income disaster occurs are given by:

$$\begin{aligned} dR_n(t) &= \frac{dS_n^{em}(t) + D(t)dt}{S_n^{em}(t)} \\ &= \Big\{\beta + \gamma\mu^I - \frac{1}{2}||\sigma^I||^2\gamma(\gamma-1)\Big\}dt + (\sigma^I)^\top dZ(t) \\ &\equiv \mu_n^{em}dt + (\sigma_n^{em})^\top dZ(t) \end{aligned}$$

and

$$\begin{aligned} dR_d(t) &= \frac{dS_d^{em}(t) + D(t)dt}{S_d^{em}(t)} \\ &= \Big\{\beta - (\hat{\delta} - \delta) + \gamma\mu^I - \frac{1}{2}||\sigma^I||^2\gamma(\gamma-1)\Big\}dt + (\sigma^I)^\top dZ(t) \\ &\equiv \mu_d^{em}dt + (\sigma_d^{em})^\top dZ(t), \end{aligned}$$

where $\mu_n^{em}$, $\mu_d^{em}$ and $\sigma_n^{em}$, $\sigma_d^{em}$ are the equilibrium equity expected return and volatility in the economy $n$ and in the economy $d$, respectively.

From the SDFs given in Theorem 4.1, we obtain before income disaster occurs that

$$\begin{aligned} dH_n(t) &= H_n(t)\Big[-\Big\{\beta + \gamma\mu^I - \frac{1}{2}\gamma(\gamma+1)||\sigma^I||^2\Big\}dt - \gamma(\sigma^I)^\top dZ(t)\Big] \\ &\equiv H_n(t)[-r_n dt - \theta_n^\top dZ(t)] \end{aligned}$$

and

$$\begin{aligned} dH_d(t) &= H_d(t)\Big[-\Big\{\beta - (\hat{\delta} - \delta) + \gamma\mu^I - \frac{1}{2}\gamma(\gamma+1)||\sigma^I||^2\Big\}dt - \gamma(\sigma^I)^\top dZ(t)\Big] \\ &\equiv H_n(t)[-r_d dt - \theta_d^\top dZ(t)], \end{aligned}$$

where $r_n$, $\theta_n$ and $r_d$, $\theta_d$ are the equilibrium risk-free interest rate and Sharpe ratio in the economy $n$ and in the economy $d$, respectively.

We now obtain the equilibrium equity premium in the economy $n$ and in the economy $d$ as follows:

$$\mu_n^{em} - r_n = \gamma||\sigma^I||^2$$

and

$$\mu_d^{em} - r_d = \gamma||\sigma^I||^2,$$

respectively, which complete the proof. **Q.E.D.**

# I Proof of Theorem 4.4

The Euler equation for the income-disaster-exposed agent's equilibrium consumption price is given by (A-40) as

$$U'(c_d(t)) = \lambda_d e^{(\beta+\delta-\hat{\delta})t} H(t),$$

where $U$ is utility over consumption and assumed to be twice continuously differentiable, strictly increasing and strictly concave. Assuming the CRRA utility preference, Itô's formula allows us to obtain the following equilibrium consumption dynamics:

$$\begin{aligned} dc_d(t) = & -\frac{1}{\gamma}\Big(\lambda_d e^{(\beta+\delta-\hat{\delta})t}H(t)\Big)^{-1/\gamma-1} d\Big(\lambda_d e^{(\beta+\delta-\hat{\delta})t}H(t)\Big) \\ & + \frac{1}{2}\frac{1}{\gamma}\Big(\frac{1}{\gamma}+1\Big)\Big(\lambda_d e^{(\beta+\delta-\hat{\delta})t}H(t)\Big)^{-1/\gamma-1}\Big\{d\Big(\lambda_d e^{(\beta+\delta-\hat{\delta})t}H(t)\Big)\Big\}^2 \\ = & -\frac{1}{\gamma}\Big(\lambda_d e^{(\beta+\delta-\hat{\delta})t}H(t)\Big)^{-1/\gamma}\Big[(\beta+\delta-\hat{\delta})dt - r_d dt - \theta^\top dZ(t)\Big] \\ & + \frac{1}{2}\frac{1}{\gamma}\Big(\frac{1}{\gamma}+1\Big)\Big(\lambda_d e^{(\beta+\delta-\hat{\delta})t}H(t)\Big)^{-1/\gamma}||\theta||^2 dt, \end{aligned}$$

where we have used the fact that

$$\begin{aligned} d\Big(\lambda_d e^{(\beta+\delta-\hat{\delta})t}H(t)\Big) = & \lambda_d e^{(\beta+\delta-\tilde{\delta}(r_d))t}H(t)\{\beta+\delta-\tilde{\delta}(r_d)\}dt \\ & + \lambda_d e^{(\beta+\delta-\tilde{\delta}(r_d))t}H(t)\{-r_d dt - \theta^\top dZ(t)\}. \end{aligned}$$

Hence,

$$\begin{aligned} \frac{dc_d(t)}{c_d(t)} = & -\frac{1}{\gamma}\Big[\{\beta+\delta-\hat{\delta}\}dt - r_d dt - \gamma(\sigma^I)^\top dZ(t)\Big] + \frac{1}{2}\frac{1}{\gamma}\Big(\frac{1}{\gamma}+1\Big)\gamma^2||\sigma^I||^2 dt \\ = & \frac{1}{\gamma}\Big(r_d - \beta + \frac{1}{2}\gamma(1+\gamma)||\sigma^I||^2 + \hat{\delta} - \delta\Big)dt + (\sigma^I)^\top dZ(t), \end{aligned}$$

where the first equality is a result of the substitution of equilibrium Sharpe ratio $\theta = \gamma\sigma^I$, which completes the proof. **Q.E.D.**

# J Proof of Theorem 4.5

Without loss of generality, we now consider income-disaster-adjusted risk-free rate $r_d + \delta$ and subjective discount factor $\beta + \delta$, thus resulting in

$$\beta_1 = r_d + \delta - \mu^I + (\sigma^I)^\top\theta$$

and

$$\beta_2 = \beta + \delta + (1-\gamma)\mu^I + \frac{1}{2}\gamma(1-\gamma)||\sigma^I||^2.$$

The partial equilibrium optimal consumption strategy given in (A-34) reduces to the following general equilibrium consumption strategy of the income-disaster-exposed agent by substituting all the equilibrium quantities in (A-34):

$$c_d(t) = (\tilde{A}(r_d) + \delta/\gamma)\Big(w_d + \frac{\xi_d I}{\beta_1}\Big), \tag{A-42}$$

where

$$
\begin{aligned}
\tilde{A}(r_d) &= \frac{\gamma-1}{\gamma}\Big(r_d + \delta - \mu^I + \gamma||\sigma^I||^2\Big) + \frac{\beta+\delta}{\gamma} + \frac{\gamma-1}{\gamma}\mu^I + \frac{1}{2}(1-\gamma)||\sigma^I||^2 \\
&= r_d + \frac{1}{\gamma}(\beta - r_d) + \delta + \frac{1}{2}(\gamma-1)||\sigma^I||^2 \\
&= r_d + \frac{1}{\gamma}\Big\{-\gamma\mu^I + \frac{1}{2}\gamma(1+\gamma)||\sigma^I||^2 + (\hat{\delta}-\delta)\Big\} + \delta + \frac{1}{2}(\gamma-1)||\sigma^I||^2 \\
&= r_d + \delta - \mu^I + \gamma||\sigma^I||^2 + \frac{\hat{\delta}-\delta}{\gamma},
\end{aligned}
$$

the first equality is a result of the substitution of the equilibrium quantities in $\beta_1$ and $\beta_3$ of $\hat{A}$ given in (A-26), and third equality is a result of the substitution of the determined equilibrium risk-free interest rate $r_d$. Hence, the equilibrium marginal propensity to consume (MPC) out of financial wealth $w_d$ is now obtained as in Theorem 4.5 by differentiating the both sides of equilibrium optimal consumption (A-42) with respect to $w_d$.

# K Effects of Borrowing Constraints on Optimal Strategies

The extent to which the agent is borrowing constrained with a range of values for $L$ affects the agent's optimal consumption (Table 4) and investment (Table 5) strategies. Tightening of borrowing by decreasing $L$ makes agents reduce their consumption amount; this response is especially significant for poor people, so their consumption smoothing is more difficult than for wealthy people. The effects of income disaster even worsens the situation for poor people. Given the significant downward jump in income in the aftermath of income disaster, the poor people who are substantially borrowing constrained would have difficulty to secure extra savings to finance their consumption needs. Hence, the consumption amount could fall further with the joint effects caused by the borrowing tightening and the income shock. Those effects also reduce the risky investment amount; this result is similar to the observation in the optimal consumption amount.

| $w \setminus L$ | $\delta = 0$ 0% | 5% | 10% | 20% | $\delta = 0.07$ 0% | 5% | 10% | 20% | $\delta = 0.08$ 0% | 5% | 10% | 20% |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 1 | 1.0922 | 1.2090 | 1.2966 | 1.4336 | 0.7261 | 0.7200 | 0.7232 | 0.7281 | 0.7077 | 0.7033 | 0.7060 | 0.7098 |
| 10 | 1.8159 | 1.8646 | 1.9104 | 1.9948 | 1.1919 | 1.1809 | 1.1808 | 1.1810 | 1.1691 | 1.1578 | 1.1575 | 1.1586 |
| 20 | 2.4041 | 2.4402 | 2.4750 | 2.5409 | 1.6679 | 1.6538 | 1.6538 | 1.6539 | 1.6406 | 1.6285 | 1.6285 | 1.6285 |
| 30 | 2.9396 | 2.9698 | 2.9991 | 3.0552 | 2.1348 | 2.1183 | 2.1185 | 2.1185 | 2.1040 | 2.0915 | 2.0918 | 2.0910 |
| 40 | 3.4507 | 3.4772 | 3.5031 | 3.5528 | 2.5966 | 2.5787 | 2.5789 | 2.5789 | 2.5636 | 2.5507 | 2.5510 | 2.5501 |
| 50 | 3.9476 | 3.9715 | 3.9949 | 4.0400 | 3.0552 | 3.0365 | 3.0368 | 3.0368 | 3.0210 | 3.0075 | 3.0078 | 3.0069 |

| $w \setminus L$ | $\delta = 0.09$ 0% | 5% | 10% | 20% | $\delta = 0.10$ 0% | 5% | 10% | 20% |
|---|---|---|---|---|---|---|---|---|
| 1 | 0.6862 | 0.6887 | 0.6909 | 0.6945 | 0.6748 | 0.6815 | 0.6784 | 0.6814 |
| 10 | 1.1396 | 1.1397 | 1.1397 | 1.1398 | 1.1223 | 1.1094 | 1.1233 | 1.1237 |
| 20 | 1.6071 | 1.6071 | 1.6072 | 1.6071 | 1.5894 | 1.5913 | 1.5890 | 1.5888 |
| 30 | 2.0680 | 2.0681 | 2.0681 | 2.0681 | 2.0498 | 2.0618 | 2.0489 | 2.0486 |
| 40 | 2.5260 | 2.5260 | 2.5260 | 2.5260 | 2.5069 | 2.5208 | 2.5059 | 2.5057 |
| 50 | 2.9820 | 2.9821 | 2.9821 | 2.9821 | 2.9620 | 2.9738 | 2.9613 | 2.9611 |

Table 4: **Optimal consumption amount for various borrowing tightening scenarios and intensity values of the large, negative shock.** Parameter values: $r = 0.02$ (risk-free rate), $\beta = 0.04$ (subjective discount rate), $\mu = 0.06$ (expected stock return), $\sigma = 0.20$ (stock volatility), $\gamma = 2$ (risk aversion), $\epsilon = 1$ (income), and $k = 0.2$ (recovery rate).

# L Technical Details behind Section 6

## L.1 Repeated income disasters

Referring to the change of measure for a general Poisson process in Chapter 11.6.1 in Shreve (2004), the SDFs in an incomplete market caused by repeated income disaster can be given in the state $i$ ($i \in \{G, B\}$) by

$$\xi^{\hat{\delta}^i,\hat{\phi}^i}(t) = \exp\Big\{ \ln\Big(\frac{\hat{\delta}^i}{\delta^i}\Big) N^G(t) - (\hat{\delta}^i - \delta^i)t \Big\} \exp\Big\{ \ln\Big(\frac{\hat{\phi}^i}{\phi^i}\Big) \mathbf{1}_{\{\tau^i = t\}} - (\hat{\phi}^i - \phi^i)t \Big\} H(t),$$

where $\hat{\delta}^i$ and $\hat{\phi}^i$ are the risk-neutral intensities to be determined.

We define the new probability measure in the state $i$ as

$$P^*(A) \equiv \int_A e^{b^i Z^\delta(t,\omega) - \frac{1}{2}(b^i)^2 t} dP(\omega) \quad \text{for all} \quad A \in \mathcal{G}$$

and the Brownian motion process under the new measure as

$$Z^*(t) \equiv Z(t) - \rho^i b^i t,$$

which is the standard Brownian motion process. The agent's dynamic wealth constraint (3) can

| | $\delta=0$ | | | | $\delta=0.07$ | | | | $\delta=0.08$ | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| $w \setminus L$ | 0% | 5% | 10% | 20% | 0% | 5% | 10% | 20% | 0% | 5% | 10% | 20% |
| 1 | 4.0426 | 6.1272 | 7.6885 | 10.1335 | 5.2955 | 5.6019 | 5.8596 | 6.2650 | 5.3110 | 5.5845 | 5.8131 | 6.1575 |
| 10 | 14.2285 | 15.0976 | 15.9148 | 17.4194 | 12.3362 | 12.2797 | 12.8383 | 12.2981 | 12.2003 | 12.1282 | 12.1269 | 12.1598 |
| 20 | 21.6927 | 22.3378 | 22.9587 | 24.1342 | 17.7237 | 17.6876 | 17.6797 | 17.6806 | 17.5918 | 17.4760 | 17.4635 | 17.5022 |
| 30 | 28.2186 | 28.7577 | 29.2809 | 30.2818 | 23.0102 | 22.9245 | 22.9209 | 22.9207 | 22.8117 | 22.6988 | 22.6947 | 22.7050 |
| 40 | 34.3084 | 34.7818 | 35.2955 | 36.1304 | 28.2267 | 28.0941 | 28.0950 | 28.0951 | 27.9683 | 27.8593 | 27.8622 | 27.8523 |
| 50 | 40.1449 | 40.5720 | 40.9892 | 41.7945 | 33.3908 | 33.2257 | 33.2293 | 33.2298 | 33.0902 | 32.9808 | 32.9875 | 32.9675 |

| | $\delta=0.09$ | | | | $\delta=0.10$ | | | |
|---|---|---|---|---|---|---|---|---|
| $w \setminus L$ | 0% | 5% | 10% | 20% | 0% | 5% | 10% | 20% |
| 1 | 5.3298 | 5.5553 | 5.7509 | 6.0713 | 5.3526 | 5.7242 | 5.7158 | 5.9980 |
| 10 | 12.0225 | 12.0263 | 12.0312 | 12.0399 | 11.8807 | 11.6225 | 11.9189 | 11.9382 |
| 20 | 17.3480 | 17.3468 | 17.3471 | 17.3471 | 17.1623 | 16.6403 | 17.2001 | 17.2086 |
| 30 | 22.5196 | 22.5197 | 22.5197 | 22.5194 | 22.3573 | 22.2349 | 22.3573 | 22.3544 |
| 40 | 27.6439 | 27.6450 | 27.6452 | 27.6454 | 27.4934 | 27.6851 | 27.4720 | 27.4644 |
| 50 | 32.7435 | 32.7451 | 32.7453 | 32.7456 | 32.5915 | 32.9486 | 32.5630 | 32.5543 |

Table 5: **Optimal investment amount for various borrowing tightening scenarios and intensity values of the large, negative shock.** Parameter values: $r = 0.02$ (risk-free rate), $\beta = 0.04$ (subjective discount rate), $\mu = 0.06$ (expected stock return), $\sigma = 0.20$ (stock volatility), $\gamma = 2$ (risk aversion), $\epsilon = 1$ (income), and $k = 0.2$ (recovery rate).

be rewritten as

$$dW(t) = \{rW(t) - c(t) + \xi I(t)\}dt + \pi(t)^\top \sigma^\top \{dZ^*(t) + (\theta + \rho^i b^i)dt\}. \tag{A-43}$$

Similar to Lemma D.3, for the fixed $\hat{\delta}^i$ and $\hat{\phi}^i$, the dynamic wealth constraint (A-43) can be converted into the following static wealth constraint:

$$E^*\Big[\int_0^\infty e^{-(\hat{\delta}^i+\hat{\phi}^i)t} H(t)\Big(c(t) - \xi I(t) + (\hat{\delta}^i + \hat{\phi}^i)W(t)\Big)dt\Big] \le w. \tag{A-44}$$

The original dynamic problem (19) can then be converted into the following static problem:

$$V^i(w, I) = \sup_{(c,W)} E^*\Big[\int_0^\infty e^{-(\beta+\delta^i+\phi^i)t}\Big(\frac{c(t)^{1-\gamma}}{1-\gamma} + \delta^i V^i\big(W(t), kI(t)\big) + \phi^i V^j\big(W(t), I(t)\big)\Big)dt\Big], \tag{A-45}$$

which is subject to (A-44).

Similar to Lemma D.4, the static optimization problem (A-45) can be reformulated as

$$V^i(w, I) = \inf_{(\lambda_i, \hat{\delta}^i, \hat{\phi}^i)} \{J^{\hat{\delta}^i,\hat{\phi}^i}(\lambda_i, I) + \lambda_i w\} = \inf_{\lambda_i}\{\inf_{\hat{\delta}^i,\hat{\phi}^i} J^{\hat{\delta}^i,\hat{\phi}^i}(\lambda_i, I) + \lambda_i w\} \equiv \inf_{\lambda_i}\{J^i(\lambda_i, I) + \lambda_i w\}, \tag{A-46}$$

where the indirect value function $J^{\hat{\delta}^i,\hat{\phi}^i}(\lambda_i, I)$ is given by

$$
\begin{aligned}
&J^{\hat{\delta}^i,\hat{\phi}^i}(\lambda_i, I)\\
&= (\xi I)^{1-\gamma}\tilde{E}\Big[\int_0^\infty e^{-(\beta_2+\delta^i+\phi^i)t}\Big\{\frac{\gamma}{1-\gamma}\Gamma^{\hat{\delta}^i,\hat{\phi}^i}(t)^{1-1/\gamma}+\Gamma^{\hat{\delta}^i,\hat{\phi}^i}(t)\Big\}dt\Big]\\
&+ (\xi I)^{1-\gamma}\tilde{E}\Big[\int_0^\infty e^{-(\beta_2+\delta^i+\phi^i)t}\Big\{\delta^i k^{1-\gamma}\Big(v_i\big(y(t)/k\big)-v_i'\big(y(t)/k\big)\frac{y(t)}{k}\Big)+\phi^i\Big(v_j\big(y(t)\big)-v_j'\big(y(t)\big)y(t)\Big)\Big\}\Big],
\end{aligned}
\tag{A-47}
$$

where $\tilde{E}$ is the expectation under the new probability measure defined as

$$\tilde{P}(A) \equiv \int_A \exp\Big(-\frac{1}{2}(1-\gamma)^2||\sigma^I||^2(t,\omega)+(1-\gamma)(\sigma^I)^\top Z^*(t,\omega)\Big)dP^*(\omega) \quad \text{for all } A\in\mathcal{G}$$

with the new Brownian motion process $\tilde{Z}$ given by

$$\tilde{Z}(t) = -(1-\gamma)\sigma^I t + \rho^i b^i t + Z^*(t),$$

$\Gamma^{\hat{\delta}^i,\hat{\phi}^i}(t)$ is a new state variable defined by

$$\Gamma^{\hat{\delta}^i,\hat{\phi}^i}(t) \equiv \lambda_i e^{(\beta+\delta^i-\hat{\delta}^i+\phi^i-\hat{\phi}^i)t}H(t)\big(\xi I(t)\big)^\gamma,$$

its dynamics are given by

$$d\Gamma^{\hat{\delta}^i,\hat{\phi}^i}(t) = \Gamma^{\hat{\delta}^i,\hat{\phi}^i}(t)\{-(\beta_1^{\hat{\delta}^i,\hat{\phi}^i}-\beta_2)dt+\beta_3^i d\tilde{Z}(t)\},$$

with

$$
\begin{aligned}
\beta_1^{\hat{\delta}^i,\hat{\phi}^i} &= r-\mu^I+(\sigma^I)^\top(\theta+\rho^i b^i)+\hat{\delta}^i-\delta^i+\hat{\phi}^i-\phi^i,\\
\beta_3^i &= \gamma(\sigma^I)^\top-(\theta+\rho^i b^i)^\top,
\end{aligned}
$$

and $v_i\big(y(t)\big)$ has the following relation with the value function $V^i\big(W(t), I(t)\big)$:

$$V^i\big(W(t), I(t)\big) = (\xi I(t))^{1-\gamma}v_i\big(y(t)\big), \quad y(t) = \frac{W(t)}{\xi I(t)}.$$

The relation (A-46) implies that

$$\big(\xi I(t)\big)^{-\gamma}v_i'\big(y(t)\big) = \lambda_i e^{(\beta+\delta^i-\hat{\delta}^i+\phi^i-\hat{\phi}^i)t}H(t)$$

and therefore,

$$v_i'(y(t)) = \Gamma^{\hat{\delta}^i,\hat{\phi}^i}(t),$$

thus demonstrating that the state variable $\Gamma^{\hat{\delta}^i,\hat{\phi}^i}(t)$ is a function of the wealth-to-income ratio $y(t)$. Hence, the indirect value function $J^{\hat{\delta}^i,\hat{\phi}^i}(\lambda_i, I)$ given in (A-47) can be restated as the following:

$$J^{\hat{\delta}^i,\hat{\phi}^i}(\lambda_i, I) \equiv (\xi I)^{1-\gamma}\varphi_i\big(z(y)\big) = J^i(\lambda_i, I)$$

with the uniquely determined the risk-neutral intensities $\hat{\delta}^i$ and $\hat{\phi}^i$ as

$$\begin{aligned}\hat{\delta}^i &= \delta^i \frac{1}{\lambda_i(y)} \frac{\partial V^i(w, kI)}{\partial w} \\ &= \delta^i \frac{1}{\lambda_i(y)} (k\xi I)^{-\gamma} v_i'(y/k) \\ &= \delta^i k^{-\gamma} \frac{v_i'(y/k)}{z(y)} \\ &= \delta^i k^{-\gamma} \frac{z(y/k)}{z(y)}\end{aligned}$$

and

$$\begin{aligned}\hat{\phi}^i &= \phi^i \frac{v_j'(y)}{z(y)} \\ &= \phi^i \frac{z_j(y)}{z(y)}\end{aligned}$$

respectively, where

$$z(y) = \lambda_i(y)(\xi I)^\gamma, \quad z_j(y) = \lambda_j(y)(\xi I)^\gamma.$$

By Feynman-Kac's formula, we obtain the following non-linear ordinary differential equation: for any $0 < z(y) < \bar{z}_i$,

$$\begin{aligned}\frac{1}{2}||\beta_3^i||^2 z(y)^2 \varphi_i''\big(z(y)\big) - (\beta_1^i - \beta_2 - \delta^i - \phi^i) z(y) \varphi_i'\big(z(y)\big) - (\beta_2 + \delta^i + \phi^i)\varphi_i\big(z(y)\big) \\ + \frac{\gamma}{1-\gamma} z(y)^{1-1/\gamma} + z(y) + \delta^i k^{1-\gamma} v_i(y/k) + \phi^i v_j(y) = 0,\end{aligned} \tag{A-48}$$

where

$$\beta_1^i = r - \mu^I + (\sigma^I)^\top(\theta + \rho^i b^i)$$

and $\bar{z}_i$ is to be determined according to the boundary conditions given by

$$\varphi'(\bar{z}_i) = \frac{L}{\xi I}, \quad \varphi''(\bar{z}_i) = 0,$$

which results from the borrowing constraint (4). After differentiating the both sides of (A-48) with respect to $z(y)$, we obtain that for any $0 < z(y) < \bar{z}_i$,

$$\begin{aligned}\frac{1}{2}||\beta_3^i||^2 z(y)^2 \varphi_i'''\big(z(y)\big) + (||\beta_3^i||^2 + \beta_2 + \delta^i + \phi^i - \beta_1^i) z(y) \varphi_i''\big(z(y)\big) - \beta_1^i \varphi_i'\big(z(y)\big) \\ - z(y)^{-1/\gamma} + 1 + \delta^i k^{-\gamma} z(y/k) \frac{y}{\partial z(y)} + \phi^i z_j(y) \frac{y}{\partial z(y)} = 0.\end{aligned} \tag{A-49}$$

We know from the relation (A-46) that

$$\varphi_i'\big(z(y)\big) = -y \quad \varphi_i''\big(z(y)\big) = -\frac{y}{\partial z(y)}$$

and denote $-\varphi_i'\big(z(y)\big)$ by $\tilde{G}_i\big(z(y)\big)$. The equation (A-49) can then be rewritten as follows: for any $0 < z(y) < \bar{z}_i$,

$$-\frac{1}{2}||\beta_3^i||^2 z(y)^2 \tilde{G}_i''\big(z(y)\big) - (||\beta_3^i||^2 + \beta_2 + \delta^i + \phi^i - \beta_1^i) z(y) \tilde{G}_i\big(z(y)\big) + \beta_1^i \tilde{G}_i\big(z(y)\big) + 1$$
$$+ \delta^i k^{-\gamma} z(y/k) \tilde{G}_i'\big(z(y)\big) + \phi^i z_j(y) \tilde{G}_i'\big(z(y)\big) = z(y)^{-1/\gamma} \tag{A-50}$$

with the boundary conditions

$$\tilde{G}_i(\bar{z}_i) = -\frac{L}{\xi I}, \quad \tilde{G}_i'(\bar{z}_i) = 0.$$

We also denote $\tilde{G}_i\big(z(y)\big) + 1/\beta_1^i$ by $G_i\big(z(y)\big)$. The equation (A-50) is then restated as follows: for any $0 < z(y) < \bar{z}_i$,

$$-\frac{1}{2}||\beta_3^i||^2 z(y)^2 G_i''\big(z(y)\big) - (||\beta_3^i||^2 + \beta_2 + \delta^i + \phi^i - \beta_1^i) z(y) G_i'\big(z(y)\big) + \beta_1^i G_i\big(z(y)\big)$$
$$+ \delta^i k^{-\gamma} z\Big(\frac{G_i\big(z(y)\big) - 1/\beta_1^i}{k}\Big) G_i'\big(z(y)\big) + \phi^i z_j(y) G_i'\big(z(y)\big) = z(y)^{-1/\gamma}. \tag{A-51}$$

For notational simplicity, we write $z(y)$ as $z$ and $z_j(y)$ as $z_j$ unless there is any confusion.

The following proposition allows us to obtain a general solution of the equation (A-51):

**Proposition L.1** *A general solution of the equation (A-51) is given by*

$$G_i(z) = \frac{1}{\hat{A}_i + (\delta^i + \phi^i)/\gamma} z^{-1/\gamma} + B_i^* z^{-\alpha_i^*} + \frac{2\phi^i z_j}{||\beta_3^i||^2(\alpha_i - \alpha_i^*)}\Big[(\alpha_i - 1) z^{-\alpha_i} \int_0^z \mu^{\alpha_i - 2} G_i(\mu) d\mu$$
$$+ (\alpha_i^* - 1) z^{-\alpha_i^*} \int_z^{\bar{z}_i} \mu^{\alpha_i^* - 2} G_i(\mu) d\mu\Big] + \frac{2\delta^i k^{1-\gamma}}{||\beta_3^i||^2(\alpha_i - \alpha_i^*)}\Big[(\alpha_i - 1) z^{-\alpha_i} \int_0^z \mu^{\alpha_i - 2} v_i\Big(\frac{G_i(\mu) - 1/\beta_1^i}{k}\Big) d\mu$$
$$+ (\alpha_i^* - 1) z^{-\alpha_i^*} \int_z^{\bar{z}_i} \mu^{\alpha_i^* - 2} v_i\Big(\frac{G_i(\mu) - 1/\beta_1^i}{k}\Big) d\mu\Big],$$

*where $B_i^*$ and $\bar{z}_i$ are the two constants to be determined by the boundary conditions:*

$$G_i(\bar{z}_i) = -\frac{L}{\xi I} + \frac{1}{\beta_1^i} \quad \text{and} \quad G_i'(\bar{z}_i) = 0, \tag{A-52}$$

*and $\alpha_i > 1$ and $-1 < \alpha_i^* < 0$ are the two roots to the following characteristic equation:*

$$F_i(\alpha; \phi^i, \delta^i) \equiv -\frac{1}{2}||\beta_3^i||^2 \alpha(\alpha - 1) + (\beta_2 + \phi^i + \delta^i - \beta_1^i)\alpha + \beta_1^i = 0.$$

**Proof.** We conjecture a general solution of the equation (A-51) as

$$G_i(z) = \frac{1}{\hat{A}_i + (\delta^i + \phi^i)/\gamma} z^{-1/\gamma} + \eta_i(z) z^{-\alpha_i} + \eta_i^*(z) z^{-\alpha_i^*}, \tag{A-53}$$

subject to

$$\eta_i'(z) z^{-\alpha_i} + (\eta_i^*(z))' z^{-\alpha_i^*} = 0,$$

where $\alpha_i > 1$ and $-1 < \alpha_i^* < 0$ are the two roots to the following characteristic equation:

$$F_i(\alpha; \phi^i, \delta^i) \equiv -\frac{1}{2}||\beta_3^i||^2 \alpha(\alpha - 1) + (\beta_2 + \phi^i + \delta^i - \beta_1^i)\alpha + \beta_1^i = 0,$$

$$\hat{A}_i = \frac{\gamma - 1}{\gamma}\Big(\beta_1^i + \frac{||\beta_3^i||^2}{2\gamma}\Big) + \frac{\beta_2}{\gamma}.$$

With direct calculations of the first and second derivative $G_i$, the first three terms of left-hand side in (A-51) become

$$\begin{aligned}
&-\frac{1}{2}||\beta_3^i||^2 z^2 G_i''(z) - (||\beta_3^i||^2 + \beta_2 - \beta_1^i + \delta^i + \phi^i) z G_i'(z) + \beta_1^i G_i(z) \\
&= z^{-1/\gamma} + \frac{||\beta_3^i||^2}{2}(\alpha_i - \alpha_i^*) z^{1-\alpha_i} \eta_i'(z) \\
&= z^{-1/\gamma} - \frac{||\beta_3^i||^2}{2}(\alpha_i - \alpha_i^*) z^{1-\alpha_i^*} (\eta_i^*(z))'.
\end{aligned}$$

Hence, the equation (A-51) reduces to the following: for $0 < z < \bar{z}_i$,

$$\frac{||\beta_3^i||^2}{2}(\alpha_i - \alpha_i^*) z^{1-\alpha_i} \eta_i'(z) = -\phi^i z_j G_i'(z) - \delta^i k^{-\gamma} z \Big(\frac{G_i(z) - 1/\beta_1^i}{k}\Big) G_i'(z) \tag{A-54}$$

and

$$\frac{||\beta_3^i||^2}{2}(\alpha_i - \alpha_i^*) z^{1-\alpha_i^*} (\eta_i^*(z))' = \phi^i z_j G_i'(z) + \delta^i k^{-\gamma} z \Big(\frac{G_i(z) - 1/\beta_1^i}{k}\Big) G_i'(z), \tag{A-55}$$

where $\bar{z}_i$ is a constant to be determined according to the boundary conditions as follows:

$$G_i(\bar{z}_i) = -\frac{L}{\xi I} + \frac{1}{\beta_1^i} \quad \text{and} \quad G_i'(\bar{z}_i) = 0.$$

Integrating the both sides of (A-54) and (A-55) from 0 to $z$ and from $z$ to $\bar{z}_i$, we get

$$\eta_i(z) = -\frac{2\phi^i z_j}{||\beta_3^i||^2(\alpha_i - \alpha_i^*)} \int_0^z \mu^{\alpha_i - 1} G_i'(\mu) d\mu - \frac{2\delta^i k^{-\gamma}}{||\beta_3^i||^2(\alpha_i - \alpha_i^*)} \int_0^z \mu^{\alpha_i - 1} \mu \Big(\frac{G_i(\mu) - 1/\beta_1^i}{k}\Big) G_i'(\mu) d\mu$$

and

$$\begin{aligned}
\eta_i^*(z) = \eta_i^*(\bar{z}_i) &- \frac{2\phi^i z_j}{||\beta_3^i||^2(\alpha_i - \alpha_i^*)} \int_z^{\bar{z}_i} \mu^{\alpha_i^* - 1} G_i'(\mu) d\mu \\
&- \frac{2\delta^i k^{-\gamma}}{||\beta_3^i||^2(\alpha_i - \alpha_i^*)} \int_z^{\bar{z}_i} \mu^{\alpha_i^* - 1} \mu \Big(\frac{G_i(\mu) - 1/\beta_1^i}{k}\Big) G_i'(\mu) d\mu.
\end{aligned}$$

Thus, the general solution (A-53) can be expressed as an integral form:

$$
\begin{aligned}
G_i(z) =& \frac{1}{\hat{A}_i + (\delta^i + \phi^i)/\gamma} z^{-1/\gamma} + \eta_i^*(\bar{z}_i) z^{-\alpha_i^*} \\
& - \frac{2\phi^i z_j}{||\beta_3^i||^2(\alpha_i - \alpha_i^*)} \Big[ z^{-\alpha_i} \int_0^z \mu^{\alpha_i - 1} G_i'(\mu) d\mu + z^{-\alpha_i^*} \int_z^{\bar{z}_i} \mu^{\alpha_i^* - 1} G_i'(\mu) d\mu \Big] \\
& - \frac{2\delta^i k^{-\gamma}}{||\beta_3^i||^2(\alpha_i - \alpha_i^*)} \Big[ z^{-\alpha_i} \int_0^z \mu^{\alpha_i - 1} \mu \Big( \frac{G_i(\mu) - 1/\beta_1^i}{k} \Big) G_i'(\mu) d\mu \\
& \qquad + z^{-\alpha_i^*} \int_z^{\bar{z}_i} \mu^{\alpha_i^* - 1} \mu \Big( \frac{G_i(\mu) - 1/\beta_1^i}{k} \Big) G_i'(\mu) d\mu \Big].
\end{aligned}
\tag{A-56}
$$

Notice that

$$
\mu \Big( \frac{G_i(\mu) - 1/\beta_1^i}{k} \Big) G_i'(\mu) = \frac{d}{d\mu} \Big\{ v_i \Big( \frac{G_i(\mu) - 1/\beta_1^i}{k} \Big) \Big\} k.
$$

Thanks to the integration by parts, the general solution (A-53) can be rewritten as

$$
\begin{aligned}
G_i(z) =& \frac{1}{\hat{A}_i + (\delta^i + \phi^i)/\gamma} z^{-1/\gamma} + B_i^* z^{-\alpha_i^*} \\
& + \frac{2\phi^i z_j}{||\beta_3^i||^2(\alpha_i - \alpha_i^*)} \Big[ (\alpha_i - 1) z^{-\alpha_i} \int_0^z \mu^{\alpha_i - 2} G_i(\mu) d\mu + (\alpha_i^* - 1) z^{-\alpha_i^*} \int_z^{\bar{z}_i} \mu^{\alpha_i^* - 2} G_i(\mu) d\mu \Big] \\
& + \frac{2\delta^i k^{1-\gamma}}{||\beta_3^i||^2(\alpha_i - \alpha_i^*)} \Big[ (\alpha_i - 1) z^{-\alpha_i} \int_0^z \mu^{\alpha_i - 2} v_i \Big( \frac{G_i(\mu) - 1/\beta_1^i}{k} \Big) d\mu \\
& \qquad + (\alpha_i^* - 1) z^{-\alpha_i^*} \int_z^{\bar{z}_i} \mu^{\alpha_i^* - 2} v_i \Big( \frac{G_i(\mu) - 1/\beta_1^i}{k} \Big) d\mu \Big],
\end{aligned}
$$

where

$$
B_i^* \equiv \eta_i^*(\bar{z}_i) - \frac{2\phi^i \bar{z}_j}{||\beta_3^i||^2(\alpha_i - \alpha_i^*)} G_i(\bar{z}_i) \bar{z}_i^{\alpha_i^* - 1} - \frac{2\delta^i k^{1-\gamma}}{||\beta_3^i||^2(\alpha_i - \alpha_i^*)} v_i \Big( \frac{G_i(\bar{z}_i) - 1/\beta_1^i}{k} \Big) \bar{z}_i^{\alpha_i^* - 1},
$$

which completes the proof. **Q.E.D.**

By the first-order condition for optimal consumption $c$ used in the derivation of the indirect value function (A-47),

$$
c = \xi I z^{-1/\gamma},
$$

as a result, the general solution given in Proposition L.1 then yields an analytic characterization of optimal consumption $c$ as follows:

$$
c(t) = \Big( \hat{A}_i + \frac{\delta^i + \phi^i}{\gamma} \Big) \Big( w + \frac{\xi I}{\beta_1^i} - \xi I B_i^* z^{-\alpha_i^*} - IP_i \Big),
$$

where $IP_i$ represents the integral parts given by

$$
IP_i = IP1_i + IP2_i, \tag{A-57}
$$

$$
\begin{aligned}
IP1_i =& \frac{2\phi^i z_j(\alpha_i - 1)\xi I z^{-\alpha_i}}{||\beta_3^i||^2(\alpha_i - \alpha_i^*)} \int_0^z \mu^{\alpha_i - 2} G_i(\mu) d\mu \\
&+ \frac{2\delta^i k^{1-\gamma}(\alpha_i - 1)\xi I z^{-\alpha_i}}{||\beta_3^i||^2(\alpha_i - \alpha_i^*)} \int_0^z \mu^{\alpha_i - 2} v_i\Big(\frac{G_i(\mu) - 1/\beta_1^i}{k}\Big) G_i'(\mu) d\mu, \\
IP2_i =& \frac{2\phi^i z_j(\alpha_i^* - 1)\xi I z^{-\alpha_i^*}}{||\beta_3^i||^2(\alpha_i - \alpha_i^*)} \int_z^{\bar{z}_i} \mu^{\alpha_i^* - 2} G_i(\mu) d\mu \\
&+ \frac{2\delta^i k^{1-\gamma}(\alpha_i^* - 1)\xi I z^{-\alpha_i^*}}{||\beta_3^i||^2(\alpha_i - \alpha_i^*)} \int_z^{\bar{z}_i} \mu^{\alpha_i^* - 2} v_i\Big(\frac{G_i(\mu) - 1/\beta_1^i}{k}\Big) G_i'(\mu) d\mu.
\end{aligned}
$$

The optimal investment $\pi$ is given by

$$
\begin{aligned}
\pi &= -\sigma^{-1}\Big\{(\theta + \rho^i b^i)(\xi I)^{-\gamma} v_i'(z) + \sigma^I\Big(-\gamma(\xi I)^{-\gamma} v_i'(z) - (\xi I)^{-\gamma} z v_i''(z)\Big)\Big\}\Big/\{(\xi I)^{-\gamma-1} v_i''(z)\} \\
&= \sigma^{-1}\{\gamma\sigma^I - (\theta + \rho^i b^i)\}\xi I \frac{v_i'(z)}{v_i''(z)} + \sigma^{-1}\sigma^I \xi I z \\
&= \sigma^{-1}\{\gamma\sigma^I - (\theta + \rho^i b^i)\}\xi I z G_i'(z) + \sigma^{-1}\sigma^I w,
\end{aligned}
$$

respectively. A direct calculation of $G_i'(z)$ is given by

$$
\begin{aligned}
G_i'(z) =& -\frac{1}{\gamma\{\hat{A}_i + (\delta^i + \phi^i)/\gamma\}} z^{-1/\gamma - 1} - \alpha_i^* B_i^* z^{-\alpha_i^* - 1} + \frac{2\phi^i z_j}{||\beta_3^i||^2 z^2} G_i(z) + \frac{2\delta^i k^{1-\gamma}}{||\beta_3^i||^2 z^2} v_i\Big(\frac{G_i(z) - 1/\beta_1^i}{k}\Big) \\
&- \frac{2\phi^i z_j}{||\beta_3^i||^2(\alpha_i - \alpha_i^*)}\Big[\alpha_i(\alpha_i - 1) z^{-\alpha_i} \int_0^z \mu^{\alpha_i - 2} G_i(\mu) d\mu + \alpha_i^*(\alpha_i^* - 1) z^{-\alpha_i^*} \int_z^{\bar{z}_i} \mu^{\alpha_i^* - 2} G_i(\mu) d\mu\Big] \\
&- \frac{2\delta^i k^{1-\gamma}}{||\beta_3^i||^2(\alpha_i - \alpha_i^*)}\Big[\alpha_i(\alpha_i - 1) z^{-\alpha_i} \int_0^z \mu^{\alpha_i - 2} v_i\Big(\frac{G_i(\mu) - 1/\beta_1^i}{k}\Big) d\mu \\
&\qquad + \alpha_i^*(\alpha_i^* - 1) z^{-\alpha_i^*} \int_z^{\bar{z}_i} \mu^{\alpha_i^* - 2} v_i\Big(\frac{G_i(\mu) - 1/\beta_1^i}{k}\Big) d\mu\Big],
\end{aligned}
$$

as a result,

$$
\begin{aligned}
\xi I z G_i'(z) =& -\frac{1}{\gamma\{\hat{A}_i + (\delta^i + \phi^i)/\gamma\}} \xi I z^{-1/\gamma} - \alpha_i^* \xi I B_i^* z^{-\alpha_i^*} \\
&+ \frac{2\phi^i z_j}{||\beta_3^i||^2 z}\Big(w + \frac{\xi I}{\beta_1^i}\Big) + \frac{2\delta^i k^{1-\gamma}}{||\beta_3^i||^2 z} \xi I v_i\Big(\frac{G_i(z) - 1/\beta_1^i}{k}\Big) \\
&- \alpha_i \times \mathrm{IP1}_i - \alpha_i^* \times \mathrm{IP2}_i.
\end{aligned}
$$

Hence, we obtain an analytic characterization of optimal investment $\pi$ as follows:

$$
\begin{aligned}
\pi(t) =& \frac{1}{\gamma}\sigma^{-1}(\theta + \rho^i b^i) w \\
&+ \frac{1}{\gamma}\sigma^{-1}(\theta + \rho^i b^i - \gamma\sigma^I)\Big[\frac{\xi I}{\beta_1^i} + (\gamma\alpha_i^* - 1)\xi I B_i^* z^{-\alpha_i^*} \\
&- \frac{2\gamma\phi^i z_j}{||\beta_3^i||^2 z}\Big(w + \frac{\xi I}{\beta_1^i}\Big) - \frac{2\gamma\delta^i k^{1-\gamma}}{||\beta_3^i||^2 z} \xi I v_i\Big(\frac{G_i(z) - 1/\beta_1^i}{k}\Big) \\
&+ (\gamma\alpha_i - 1) \times \mathrm{IP1}_i + (\gamma\alpha_i^* - 1) \times \mathrm{IP2}_i\Big].
\end{aligned}
$$

The proofs of the theorems for the general equilibrium quantities and the equilibrium MPC are straightforward by following the proofs of Theorem 4.3 and Theorem 4.5.

## L.2 Uninsurable diffusive and continuous income shocks

Following Karatzas *et al.* (1991), the SDFs in an incomplete market caused by uninsurable diffusive and continuous income shocks can be given by

$$dH^{\zeta}(t) = -H^{\zeta}(t)\{rdt + \theta^{\top} dZ(t) + \zeta dZ^{I}(t)\}, \quad H^{\zeta}(0) = 1,$$

where $\zeta$ is the market price of uninsurable income shocks to be determined.

Similar to Lemma D.3, for a fixed $\zeta$, the dynamic wealth constraint (3) can be converted into the following static wealth constraint:

$$E\Big[\int_0^{\infty} H^{\zeta}(t)\Big(c(t) - \xi I(t)\Big)dt\Big] \le w. \tag{A-58}$$

We then need to solve the following static optimization problem:

$$V(w, I) \equiv \sup_{c} E\Big[\int_0^{\infty} e^{-\beta t}\frac{c(t)^{1-\gamma}}{1-\gamma}dt\Big], \tag{A-59}$$

subject to the static wealth constraint (A-58).

Similar to Lemma D.4, the static optimization problem (A-59) can be reformulated as

$$V(w, I) = \inf_{\lambda,\zeta}\{J^{\zeta}(\lambda, I) + \lambda w\} = \inf_{\lambda}\{\inf_{\zeta} J^{\zeta}(\lambda, I) + \lambda w\} \equiv \inf_{\lambda}\{J(\lambda, I) + \lambda w\},$$

where the indirect value function $J^{\zeta}(\lambda, I)$ is given by

$$\begin{aligned} J^{\zeta}(\lambda, I) &= (\xi I)^{1-\gamma}\tilde{E}\Big[\int_0^{\infty} e^{-\beta_2 t}\Big\{\frac{\gamma}{1-\gamma}\Gamma^{\zeta}(t)^{1-1/\gamma} + \Gamma^{\zeta}(t)\Big\}dt\Big] \\ &= (\xi I)^{1-\gamma}\varphi_{\zeta}(z) \end{aligned}$$

with $z = \lambda(\xi I)^{\gamma}$, where $\tilde{E}$ is the expectation under the new probability measure defined as

$$\tilde{P}(A) \equiv \int_A \exp\Big(-\frac{1}{2}(1-\gamma)^2(\sigma^I)^2(t,\omega) + (1-\gamma)\sigma^I\rho^{\top}Z(t,\omega) + (1-\gamma)\sigma^I\sqrt{1-||\rho||^2}Z^I(t,\omega)\Big)dP(\omega) \text{ for all } A \in \mathcal{F}$$

with the new Brownian motion processes $\tilde{Z}$ and $\tilde{Z}^I$ given by

$$\tilde{Z}(t) = -(1-\gamma)\sigma^I\rho^{\top}t + Z(t) \text{ and } \tilde{Z}^I(t) = -(1-\gamma)\sigma^I\sqrt{1-||\rho||^2}t + Z^I(t),$$

$$\Gamma^\zeta(t) \equiv \lambda e^{\beta t} H^\zeta(t)(\xi I(t))^\gamma,$$

and its dynamics are given by

$$d\Gamma^\zeta(t) = \Gamma^\zeta(t)\Big\{ -(\tilde{\beta}_1^\zeta - \beta_2)dt + \tilde{\beta}_3 d\tilde{Z}(t) + (\gamma\sigma^I\sqrt{1-||\rho||^2} - \zeta)d\tilde{Z}^I(t)\Big\}$$

with

$$\tilde{\beta}_1^\zeta = \tilde{\beta}_1 + \zeta\sigma^I\sqrt{1-||\rho||^2}, \quad \tilde{\beta}_1 = r - \mu^I + \sigma^I\rho^\top\theta,$$
$$\tilde{\beta}_3 = \gamma\sigma^I\rho^\top - \theta^\top,$$

and $\mathcal{F} = \{\mathcal{F}_t; t \geq 0\}$ is the standard $P$-augmentation of $\sigma\big(Z(s); 0 \leq s \leq t\big)$ and $\sigma\big(Z^I(s); 0 \leq s \leq t\big)$ generated by Brownian motion processes $Z(t)$ and $Z^I(t)$, respectively.

By Feynman-Kac's formula, we obtain the following non-linear ordinary differential equation: for any $0 < z < \bar{z}$,

$$\inf_\zeta \Big[\frac{1}{2}\Big\{||\tilde{\beta}_3||^2 + (\gamma\sigma^I\sqrt{1-||\rho||^2} - \zeta)^2\Big\} z^2\varphi_\zeta''(z) - (\tilde{\beta}_1^\zeta - \beta_2)z\varphi_\zeta'(z) - \beta_2\varphi_\zeta(z) + \frac{\gamma}{1-\gamma}z^{1-1/\gamma} + z\Big] = 0, \tag{A-60}$$

where $\bar{z}$ is to be determined according to the boundary conditions given by

$$\varphi_\zeta'(\bar{z}) = \frac{L}{\xi I}, \quad \varphi_\zeta''(\bar{z}) = 0.$$

Applying the first-order condition for $\zeta$ results in

$$\zeta = \gamma\sigma^I\sqrt{1-||\rho||^2} + \sigma^I\sqrt{1-||\rho||^2}\frac{\varphi_\zeta'(z)}{z\varphi_\zeta''(z)}. \tag{A-61}$$

With substitution of the above first-order condition in the equation (A-60), the equation is restated as

$$\frac{1}{2}||\tilde{\beta}_3||^2 z^2\varphi_\zeta''(z) - (\tilde{\beta}_1 - \beta_2)z\varphi_\zeta'(z) - \beta_2\varphi_\zeta(z) + \frac{\gamma}{1-\gamma}z^{1-1/\gamma} + z - \frac{1}{2}(\sigma^I)^2\frac{\varphi_\zeta'(z)^2}{\varphi_\zeta''(z)} = 0, \quad 0 < z < \bar{z}. \tag{A-62}$$

Denote $-\varphi_\zeta'(z)$ by $\tilde{G}(z)$. By differentiating the both sides of (A-62) with respect to $z$, the equation (A-62) can then be rewritten as

$$-\frac{1}{2}||\tilde{\beta}_3||^2 z^2\tilde{G}''(z) - (||\tilde{\beta}_3||^2 + \beta_2 - \tilde{\beta}_1)z\tilde{G}'(z) + \tilde{\beta}_1\tilde{G}(z) + 1 - \frac{1}{2}(\sigma^I)^2\frac{d}{dz}\Big(\frac{\tilde{G}(z)^2}{\tilde{G}'(z)}\Big) = 0, \quad 0 < z < \bar{z}, \tag{A-63}$$

with the boundary conditions

$$\tilde{G}(\bar{z}) = -\frac{L}{\xi I}, \quad \tilde{G}'(\bar{z}) = 0.$$

Denote also $\tilde{G}(z) + 1/\tilde{\beta}_1$ by $G(z)$. The equation (A-63) is therefore restated as

$$-\frac{1}{2}||\tilde{\beta}_3||^2 z^2 G''(z) - (||\tilde{\beta}_3||^2 + \beta_2 - \tilde{\beta}_1) z G'(z) + \tilde{\beta}_1 G(z) - \frac{1}{2}(\sigma^I)^2 \frac{d}{dz}\Big(\frac{(G(z) - 1/\tilde{\beta}_1)^2}{G(z)}\Big) = 0, \;\; 0 < z < \bar{z}, \tag{A-64}$$

with the boundary conditions

$$G(\bar{z}) = -\frac{L}{\xi I} + \frac{1}{\tilde{\beta}_1}, \quad G'(\bar{z}) = 0. \tag{A-65}$$

The market price of uninsurable income shocks $\zeta$ is now obtained from (A-61) by $G(z)$ as

$$\zeta = \gamma \sigma^I \sqrt{1 - ||\rho||^2} + \sigma^I \sqrt{1 - ||\rho||^2} \frac{G(z) - 1/\tilde{\beta}_1}{z G'(z)}.$$

Similar to Proposition E.1 and Proposition L.1, a general solution of the equation (A-63) is given by

$$\begin{aligned} G(z) = & \frac{1}{\tilde{A}} z^{-1/\gamma} + \tilde{B}^* z^{-\tilde{\alpha}^*} \\ & + \frac{(\sigma^I)^2 (1 - ||\rho||^2)}{||\tilde{\beta}_3||^2 (\tilde{\alpha} - \tilde{\alpha}^*)} \Big[ (\tilde{\alpha} - 1) z^{-\tilde{\alpha}} \int_0^z \mu^{\tilde{\alpha} - 2} \Big( \frac{\big(G(\mu) - 1/\tilde{\beta}_1\big)^2}{G'(\mu)} \Big) d\mu \\ & + (\tilde{\alpha}^* - 1) z^{-\tilde{\alpha}^*} \int_z^{\bar{z}} \mu^{\tilde{\alpha}^* - 2} \Big( \frac{\big(G(\mu) - 1/\tilde{\beta}_1\big)^2}{G'(\mu)} \Big) d\mu \Big], \end{aligned}$$

where $\tilde{B}^*$ and $\bar{z}$ are the two constants to be determined by the boundary conditions given in (A-65), and $\tilde{\alpha} > 1$ and $-1 < \tilde{\alpha}^* < 0$ are the two roots of the following characteristic equation:

$$\tilde{F}(\alpha) \equiv -\frac{1}{2}||\tilde{\beta}_3||^2 \alpha(\alpha - 1) + (\beta_2 - \tilde{\beta}_1)\alpha + \tilde{\beta}_1 = 0.$$

# Additional Reference


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