TRADABLE FACTOR RISK PREMIA AND ORACLE TESTS OF ASSET PRICING MODELS

Alberto Quaini **Fabio Trojani** Ming Yuan

Financial Econometrics Conference, Janaway Institute, Cambridge, June 2024

Factor asset pricing models

- Two major objectives:

1. Determine which factors price a given cross-section of asset returns

   $\longrightarrow$ Which factors enter the SDF

2. Quantify the compensations for exposures to factor risk $\longleftarrow$ this paper

Relevant facts about this literature

- A zoo of factors and a zoo of estimation methods

- Massive fragility of conventional approaches:

  1. Misspecification
     - Most factor models are still low-dimensional
     - Low-dimensional models are misspecified for a large enough cross-section

  2. Identification
     - Methods rely on a full rank covariance between factors and returns
     - Widely known to be violated in practice: some complicated solutions are available

$\longrightarrow$ Need methods that do not require exact factor spanning and full covariance rank

Factor asset pricing models

- Conventional inference approaches for factor risk premia:

  (Sharpe 64, Lintner 65, Fama MacBeth 73, Hansen 82, Chen Roll Ross 86, Fama French 93, Shanken 92, Jagannathan Wang 96, Lustig Verdelhan 07, Savov 11, ...)

$\longrightarrow$ Project asset expected returns on factor exposures $\boldsymbol{\beta} = \boldsymbol{V}_{RF}\boldsymbol{V}_F^{-1}$:

$$\boldsymbol{\lambda} = \text{Proj}(\boldsymbol{\mu}_R|\boldsymbol{\beta}) = -\mathbb{C}ov\left[\boldsymbol{F}, M(\boldsymbol{F})\right], \qquad M(\boldsymbol{F}) = \text{some candidate SDF}$$

  - Which SDF $M(\boldsymbol{F})$? Is it correctly specified? If not, how to weight pricing errors?
  - Identified if and only if $\boldsymbol{\beta}$ has full column rank

Useless/weak factors

- The presence of weak or useless factors, i.e., poorly or not correlated with asset returns:
  - Undermines identification of risk premia as expected return projections on factor betas (Kan Zhang 99, Kleibergen 09, Gospodinov Kan Robotti 14, 17, ...)
  - Non standard asymptotic properties: conservative inference on strong factors and spuriously liberal inference on useless/weak factors (Kleibergen 09, Gospodinov Kan Robotti 14, ...)
  - Complex interaction with potential asset pricing model misspecification (Kleibergen Zhan 2023)

- A ZOO of potentially data-mined factors (Harvey Liu Zhu 16, ...) exacerbates the issue

Selected literature

- Kleibergen 09, Kleibergen Zhang 20, 22: Tests for risk premia robust to weak identification
- Kleibergen Zhang 18: Identification robust tests for risk premia of factor mimicking portfolios
- Gospodinov Kan Robotti 14: Conservative sequential factor selection and inference procedure
- Bryzgalova 15: Oracle two-step penalization based estimation ad inference
- Anatolyev Mikusheva 22: Sample-splitting instrumental variables approach
- Bryzgalova Julliard Huang 23: Tractable Bayesian framework for analyzing the factor zoo

Our approach: Estimate risk premia

1. Without assuming SDF is fully spanned by given factors
2. Without relying on a full rank of factor-return betas
3. Staying rooted in economic theory with clear and intuitive interpretation

This paper: theory

- We rely on tradable factor risk premia and show their many desirable properties
  - Clear economic interpretation and direct link to expected returns of tradable component
  - Do not require factors to fully span the SDF
  - Remain identified even in reduced-rank models
  - Sample estimators still suffer the presence of weak factors
  - We build a closed-form Oracle estimator which:

    $\longrightarrow$ consistently removes weak/useless factors

    $\longrightarrow$ allows for "uniform" inference on the remaining factor risk premia

This paper: application

- Provide a framework for studying the factor zoo with useless/weak factors
- Search for a robust subset of economically relevant, well-identified models
- Explore factor space dimension and factor composition of well-identified models
  - First tier: MKT, SMB (Fama French 1992), MGMT (Stambaugh Yuan 2017), ICR (He Kelly Manela 2017) and MKT* (Daniel et al. 2020)
  - Second tier: BEH_FIN (Daniel et al. 2020), COMP_ISSUE (Daniel Titman 2006) and CMA (Fama French 2015)
- Document the fragility of other established factor risk premium definitions

TRADABLE FACTOR RISK PREMIA

Tradable factor risk premia

Tradable factor risk premia: Negative factor covariance with the minimum variance SDF projection on returns $M_R = 1 - \boldsymbol{\mu}_R \boldsymbol{V}_R^{-1}(\boldsymbol{R} - \boldsymbol{\mu}_R)$:

$$\boldsymbol{\lambda}^* := -\mathbb{C}ov(\boldsymbol{F}, M_R) = \boldsymbol{V}_{FR}\boldsymbol{V}_R^{-1}\boldsymbol{\mu}_R$$

- **Specification**: Factors are not required to span the SDF
- **Identification**: Well-defined if $\boldsymbol{V}_R$ is invertible, independently of the rank of $\boldsymbol{V}_{FR}$
- **Interpretation**: Equals risk premium of (tradable) factor projection on returns (Balduzzi Kallal 97)
  - In full-rank models, equals 2-pass risk premium of factor mimicking portfolio (Huberman Kandel Stambaugh 87, Breeden Gibbson Litzenberger 89,...)

Other desirable properties

- **Separability**: Tradable risk premium of a factor is unaffected by the other factors

$$\lambda_k^* = -\mathbb{C}ov[F_k, M_R] = \mathbb{C}ov[F_k, \boldsymbol{R}]\boldsymbol{V}_R^{-1}\boldsymbol{\mu}_R$$

- Assigns a zero risk premium to any useless/weak factor: $\mathbb{C}ov[F_k, \boldsymbol{R}] = 0 \implies \lambda_k^* = 0$

- Equals factor's expectation of any factor in the span of test assets returns

$$F_k = \boldsymbol{\gamma}'\boldsymbol{R} \qquad \implies \qquad \lambda_k^* = \boldsymbol{\gamma}'\boldsymbol{\mu}_R$$

- Invariant to simple repackagings of test assets (Kandel Stambaugh '95)

Relation to misspecification-robust risk premia

- Misspecification-robust factor risk premia (Kan Robotti Shanken (2013)):

$$\boldsymbol{\lambda}(\boldsymbol{W}) \in \underset{\boldsymbol{\lambda}}{\operatorname{argmin}}\ (\boldsymbol{\mu}_R - \boldsymbol{\beta}\boldsymbol{\lambda})'\boldsymbol{W}(\boldsymbol{\mu}_R - \boldsymbol{\beta}\boldsymbol{\lambda}) = \{\boldsymbol{\lambda} : \boldsymbol{\beta}'\boldsymbol{W}\boldsymbol{\beta}\boldsymbol{\lambda} = \boldsymbol{\beta}'\boldsymbol{W}\boldsymbol{\mu}_R\}.$$

- Identified if and only if $\boldsymbol{\beta}$ (equivalently $\boldsymbol{V}_{RF}$) has full column rank

- In full-rank models:
  - Non-tradable factors: $\boldsymbol{\lambda}(\boldsymbol{V}_R^{-1}) = \boldsymbol{V}_F(\boldsymbol{V}_{FR}\boldsymbol{V}_R^{-1}\boldsymbol{V}_{RF})^{-1}\boldsymbol{\lambda}^*$
  - Tradable factors: $\boldsymbol{\lambda}(\boldsymbol{V}_R^{-1}) = \boldsymbol{\lambda}^*$

- TFRP allow for misspecified and reduced-rank models

ESTIMATION OF TRADABLE FACTOR RISK PREMIA

Estimators of tradable factor risk premia

- Sample estimator:

$$\hat{\boldsymbol{\lambda}}^* := \widehat{\boldsymbol{V}}_{FR}\widehat{\boldsymbol{V}}_R^{-1}\hat{\boldsymbol{\mu}}_R = \frac{1}{T}\sum_{t=1}^{T}\widehat{\boldsymbol{V}}_{FR}\widehat{\boldsymbol{V}}_R^{-1}\boldsymbol{F}_t$$

- Oracle estimator (robust to weak factors): $\check{\boldsymbol{\lambda}}^* := \left[\text{hard-threshold}(\hat{\lambda}_k^*; \tau_T w_k)\right]_{k=1}^{K}$ where

$$\text{hard-threshold}(\hat{\boldsymbol{\lambda}}_k^*; \tau_T w_k) = \begin{cases} \hat{\lambda}_k^* & |\hat{\lambda}_k^*| > \tau_T w_k \\ 0 & \text{else} \end{cases}$$

  - penalty parameter $\tau_T > 0$ and adaptive weights $w_k = 1/\left\|\widehat{\mathbb{C}or}[F_k, \boldsymbol{R}]\right\|_2^2$
  - Equivalent to penalized minimum distance correction of $\hat{\boldsymbol{\lambda}}^*$

Assumptions

- $(\boldsymbol{R}_t, \boldsymbol{F}_t)$ near-epoch dependent with finite fourth moments
- $\boldsymbol{V}_R$ is invertible
- Joint covariance of factors and returns:

$$\boldsymbol{V}^{(T)} := \begin{bmatrix} \boldsymbol{V}_R & \boldsymbol{V}_{RF}^{(T)} \\ \boldsymbol{V}_{FR}^{(T)} & \boldsymbol{V}_F \end{bmatrix}, \quad \boldsymbol{V}_{FR}^{(T)} := \boldsymbol{V}_{FR} + \boldsymbol{\Delta}/\sqrt{T}$$

  - $\boldsymbol{V}_{FR}$ has some rank $0 \leq \text{Rank}(\boldsymbol{V}_{FR}) \leq K$
  - $\boldsymbol{\Delta}$ models a sequence of local tradable risk premia distortions of size $1/\sqrt{T}$:

$$\boldsymbol{\lambda}^{*(T)} := \boldsymbol{V}_{FR}^{(T)} \boldsymbol{V}_R^{-1} \boldsymbol{\mu}_R = \boldsymbol{V}_{FR} \boldsymbol{V}_R^{-1} \boldsymbol{\mu}_R + \frac{1}{\sqrt{T}} \boldsymbol{\Delta} \boldsymbol{V}_R^{-1} \boldsymbol{\mu}_R$$

Asymptotic behavior of $\hat{\lambda}^*$ with useless factors

Theorem

If $\boldsymbol{\Delta} = \mathbf{0}$, i.e., there are no local distortions:

$$\sqrt{T}(\hat{\boldsymbol{\lambda}}^* - \boldsymbol{\lambda}^*) \to_d \mathcal{N}(\mathbf{0}, \boldsymbol{\Sigma}), \quad \boldsymbol{\Sigma} := \lim_{T\to\infty} \frac{1}{T} \sum_{t,s=1}^{T} \mathbb{E}(\boldsymbol{h}_t \boldsymbol{h}_s')$$

$$\boldsymbol{h}_t := \bar{\boldsymbol{F}}_t \bar{\boldsymbol{R}}_t' \boldsymbol{V}_R^{-1} \boldsymbol{\mu}_R - \boldsymbol{V}_{FR} \boldsymbol{V}_R^{-1} \bar{\boldsymbol{R}}_t \bar{\boldsymbol{R}}_t' \boldsymbol{V}_R^{-1} \boldsymbol{\mu}_R + \boldsymbol{V}_{FR} \boldsymbol{V}_R^{-1} \bar{\boldsymbol{R}}_t$$

$$(\bar{\boldsymbol{R}}_t := \boldsymbol{R}_t - \boldsymbol{\mu}_R \ , \ \bar{\boldsymbol{F}}_t := \boldsymbol{F}_t - \boldsymbol{\mu}_F)$$

Asymptotic behavior of $\hat{\lambda}^*$ under local distortions

Theorem

$$\sqrt{T}(\hat{\boldsymbol{\lambda}}^* - \boldsymbol{\lambda}^*) \to_d \mathcal{N}(\boldsymbol{\eta}(\boldsymbol{\Delta}), \boldsymbol{\Sigma}(\boldsymbol{\Delta}))$$

$$\boldsymbol{\eta}(\boldsymbol{\Delta}) = \boldsymbol{\Delta} \boldsymbol{V}_R^{-1} \boldsymbol{\mu}_R$$

$$\boldsymbol{\Sigma}(\boldsymbol{\Delta}) = \boldsymbol{\Sigma} - \boldsymbol{\eta}(\boldsymbol{\Delta})\boldsymbol{\eta}'(\boldsymbol{\Delta})$$

- If there are no local distortions in the strong factor risk premia ($\boldsymbol{\Delta}_{\mathcal{S}} \boldsymbol{V}_R^{-1} \boldsymbol{\mu}_R = \mathbf{0}_{\mathcal{S}}$), their asymptotic distribution is not impacted by $\boldsymbol{\Delta}$
- No impact on the estimator's mean square error

Asymptotic behavior of $\check{\lambda}^*$ with useless/weak factors

Theorem

Let tuning parameter $\tau_T\sqrt{T} \to 0$ and $\tau_T T \to \infty$. If there are no local distortions in the strong factor risk premia,

Oracle factor selection

$$\Pr(\check{\mathcal{S}} = \mathcal{S}) \to 1, \qquad \text{where} \qquad \begin{cases} \mathcal{S} := \{k \ : \ \mathbb{C}or[F_k, \boldsymbol{R}] \neq 0\} \\ \check{\mathcal{S}} := \{k \ : \ \check{\lambda}_k^* \neq 0\} \end{cases}$$

Oracle asymptotic distribution

$$\sqrt{T}(\check{\boldsymbol{\lambda}}_{\mathcal{S}}^* - \boldsymbol{\lambda}_{\mathcal{S}}^*) \to_d \mathcal{N}(\mathbf{0}, \boldsymbol{\Sigma}_{\mathcal{S}}), \quad \boldsymbol{\Sigma}_{\mathcal{S}} := \lim_{T\to\infty} \frac{1}{T} \sum_{t,m=1}^{T} \mathbb{E}(\boldsymbol{h}_{\mathcal{S}t} \boldsymbol{h}_{\mathcal{S}m}')$$

"Uniform" Oracle distribution under local distortions

Proposition

Oracle factor selection

$$\Pr(\check{\mathcal{S}} = \mathcal{S}) \to 1, \qquad \text{where} \qquad \begin{cases} \mathcal{S} := \{k \in \{1,\ldots,K\} \; : \; \mathbb{C}or[F_k, \boldsymbol{R}] \neq 0\} \\ \check{\mathcal{S}} := \{k \in \{1,\ldots,K\} \; : \; \check{\lambda}_k^* \neq 0\} \end{cases}$$

Oracle asymptotic distribution

$$\sqrt{T}(\check{\boldsymbol{\lambda}}_{\mathcal{S}}^* - \boldsymbol{\lambda}_{\mathcal{S}}^*) \to_d \mathcal{N}(\boldsymbol{\eta}_{\mathcal{S}}(\boldsymbol{\Delta}), \boldsymbol{\Sigma}_{\mathcal{S}} - \boldsymbol{\eta}_{\mathcal{S}}(\boldsymbol{\Delta})\boldsymbol{\eta}_{\mathcal{S}}'(\boldsymbol{\Delta})), \qquad \boldsymbol{\eta}_{\mathcal{S}}(\boldsymbol{\Delta}) = \boldsymbol{\Delta}_{\mathcal{S}} \boldsymbol{V}_R^{-1} \boldsymbol{\mu}_R$$

- Consistent removal of weak/useless factors
- "Uniform" convergence in distribution over local distortions for strong factor risk premia

MONTE CARLO STUDY

▶ Skip simulation

Monte Carlo simulation setting

Simulation of returns and factors with moments calibrated from data:

$$\begin{pmatrix} \boldsymbol{R}_t \\ \boldsymbol{F}_t \end{pmatrix} \sim ii\mathcal{N}\left(\begin{pmatrix} \boldsymbol{\mu}_R \\ \boldsymbol{\mu}_F \end{pmatrix}, \begin{bmatrix} \boldsymbol{V}_R & \boldsymbol{V}_{RF} \\ \boldsymbol{V}_{FR} & \boldsymbol{V}_F \end{bmatrix}\right)$$

42 test assets: 25 ME-BTM + 17 IND

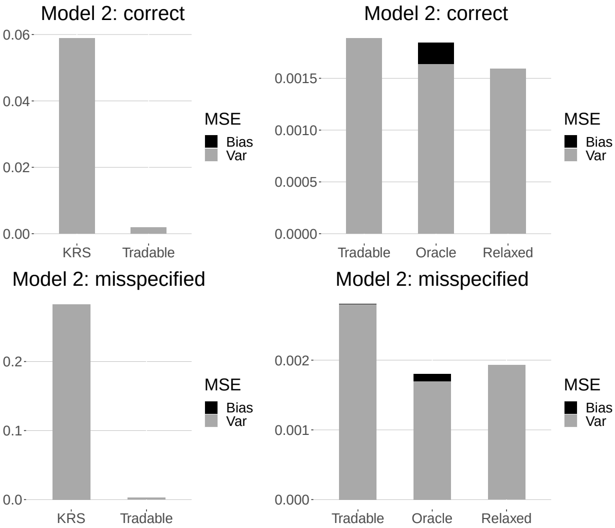
15 factors: 5FF, Mom, q-factor Hou et al. 2015, intermediary factor He et al. (2017), and betting-against beta factor Frazzini and Pedersen (2014)

Correctly specified and misspecified models:

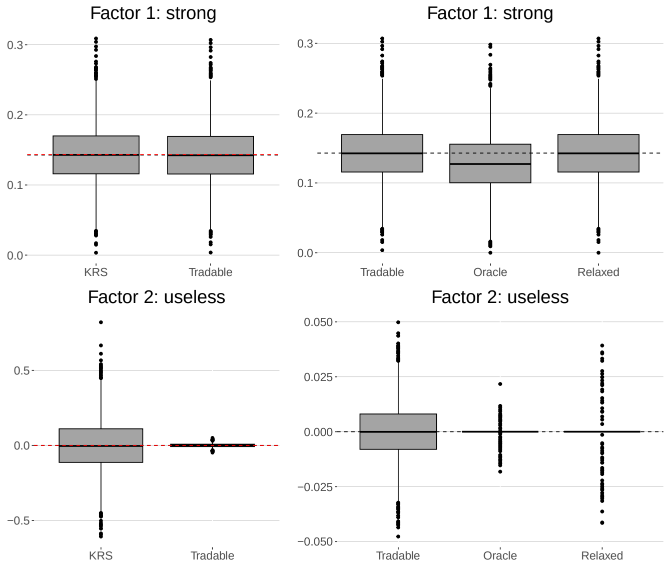
1 S || 1 S, 2 U || 1 S, 2 W || 3 S, 11 U, 1 W

S: strong, U: useless, W: weak

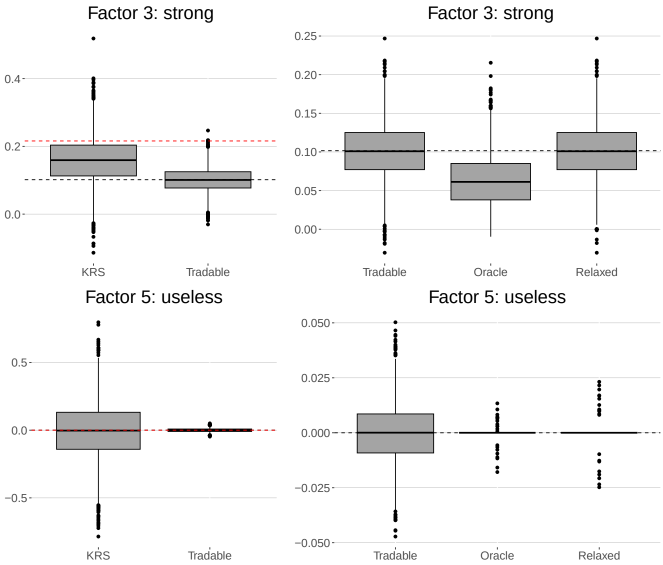
MSE and bias-variance decomposition: Model 2 (1 S, 2 U)



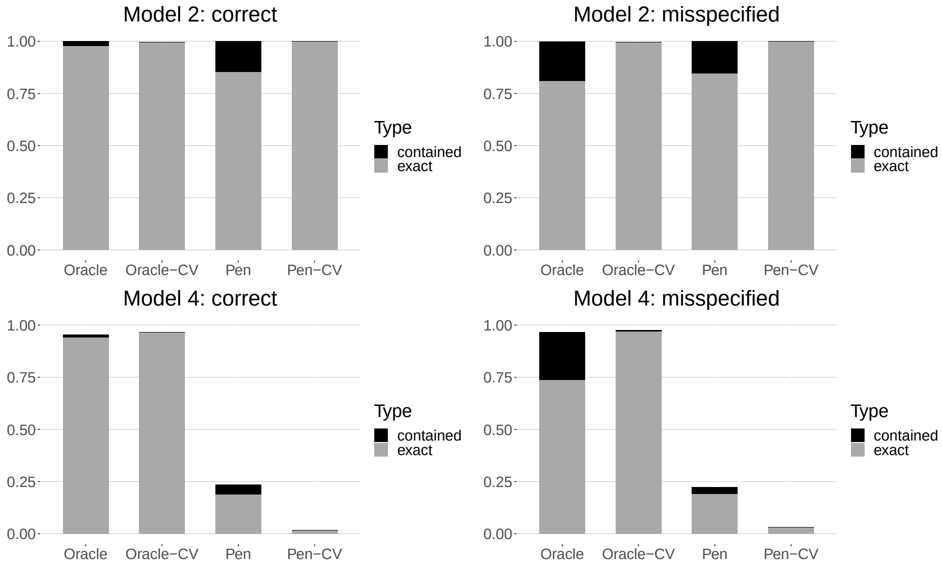
Point estimates distribution: Model 2 (1 S, 2 U) correct



Point estimates distribution: Model 4 (3 S, 11 U, 1 W) correct



Factor selection: contained $\Pr(\mathcal{S} \subset \check{\mathcal{S}})$ and exact $\Pr(\mathcal{S} = \check{\mathcal{S}})$



EMPIRICAL ANALYSIS

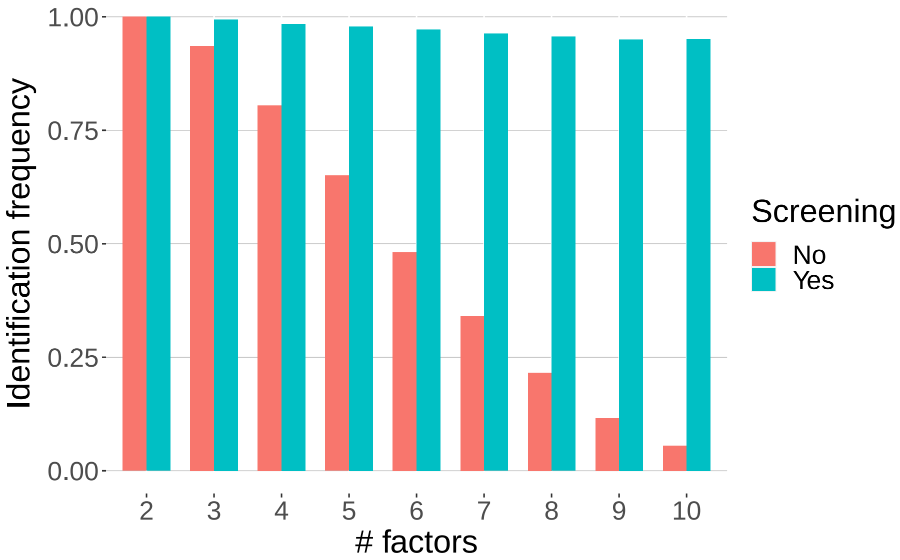
Properties of empirical asset pricing models from the factor zoo

- A diagnostics framework for studying factor asset pricing models from the zoo
- Ideally detect a robust subset of economically relevant, well-identified models
- Explore factor space dimension and factor composition of well-identified models
- Quantify role of weak identification/misspecification for interpreting factor risk premia

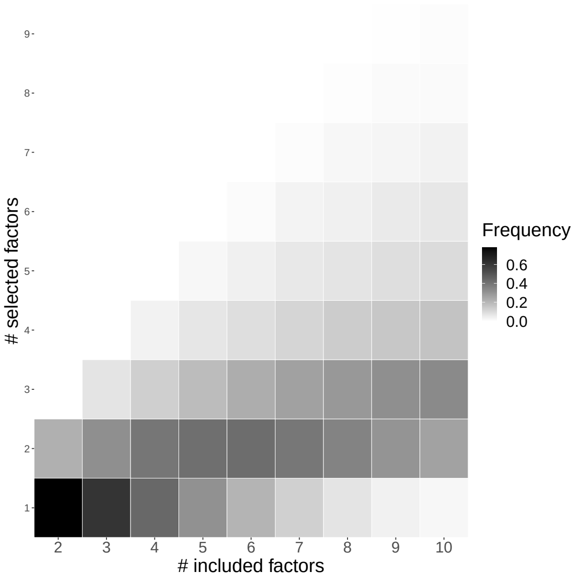
Empirical setting

- 51 factors from Bryzgalova Julliard Huang 2023 and test assets 25 ME/BTM + 17 IND
- Randomized models with 1–10 factors (always including the MKT)
- Study model properties before and after Oracle factor screening
- Test model identification via $\beta-$rank test in Kleibergen Paap 2006 and Chen Fang 2019

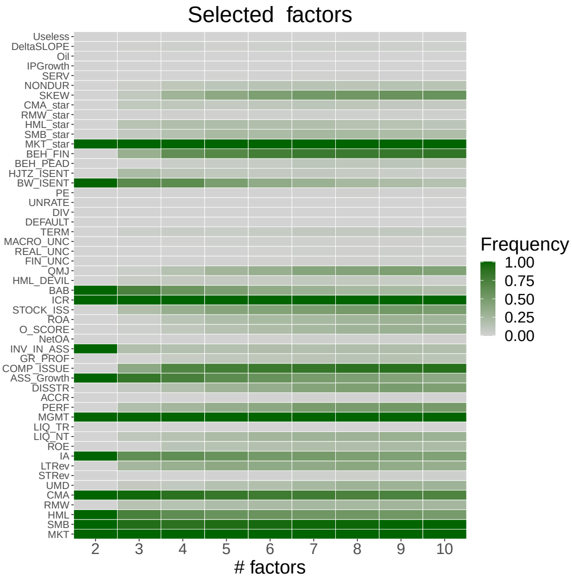
Model identification frequency



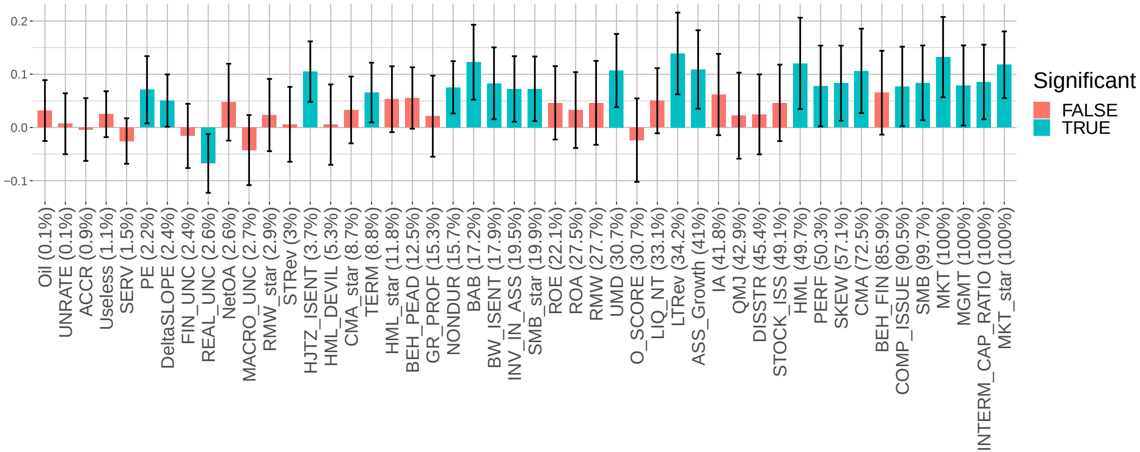
Post-screening factor space dimension



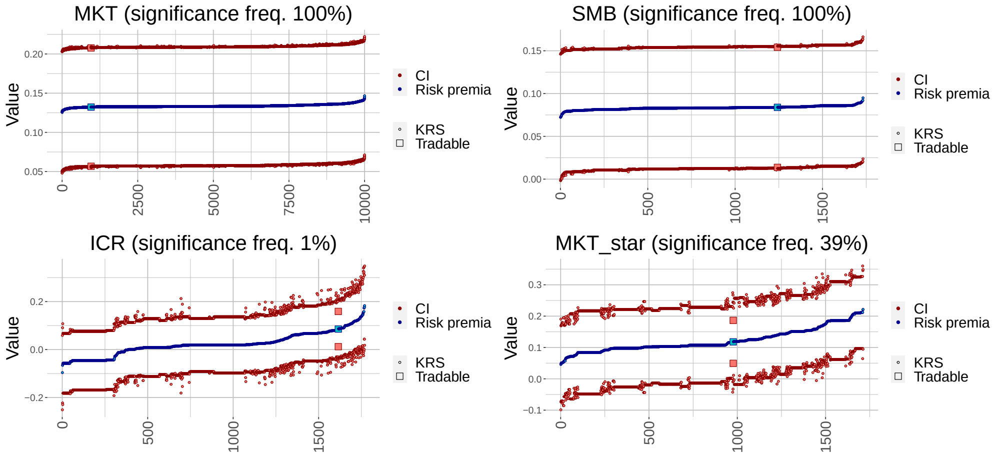
Factor selection frequency



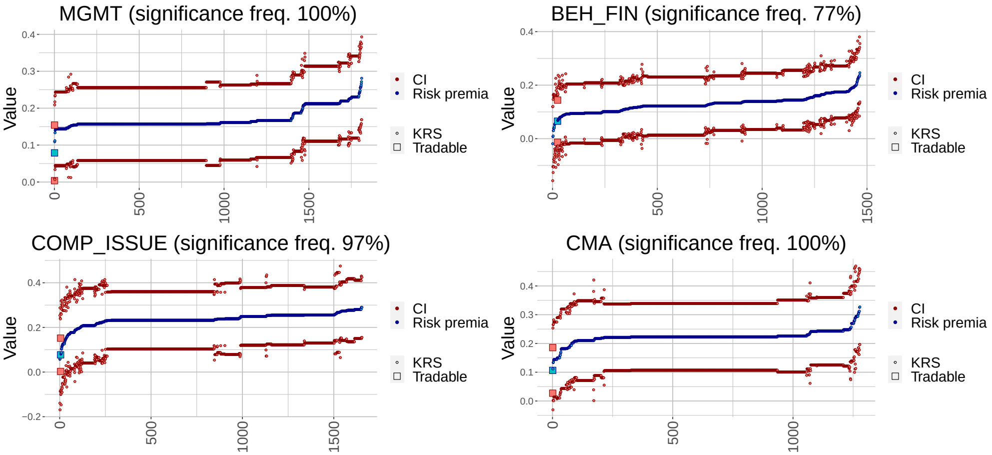
Tradable factor risk premia



Selection of misspecification-robust factor risk premia



Selection of misspecification-robust factor risk premia



Conclusions

- Tradable factor risk premia are robust to useless/weak factors and have desirable properties
- Oracle estimators have "uniform" asymptotic distributions even with weak factors
- Allow to diagnose/screen the factor zoo; Standard (conditional) analysis can follow
- Post-screening evidence points to low-dimensional, well-identified but misspecified models
- Interpretation of other established notions of factor risk premia is a challenge
- Software: R package intrinsicFRP available on CRAN

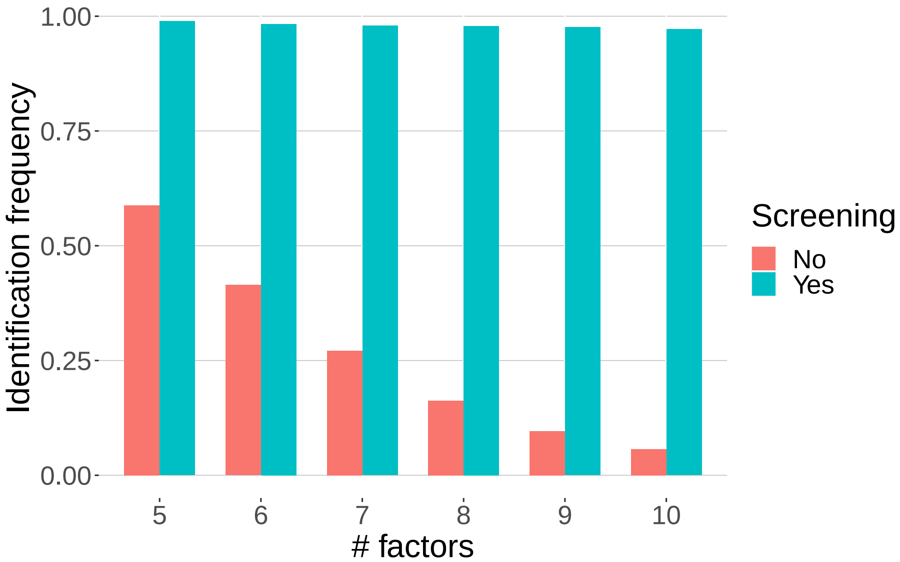
THANK YOU!

APPENDIX

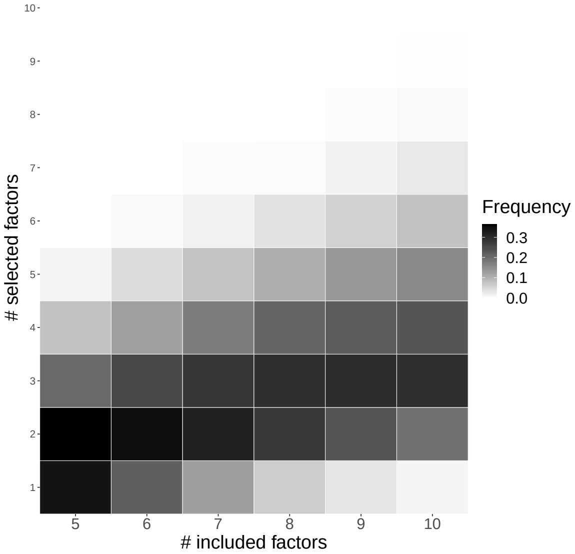
Robustness check

- Test assets: 25 size/book-to-market and 8 PCs
- PCs are extracted from 17 industry and 310 double-sorted portfolios
  - Portfolios sorted on: size, book-to-market, operating profitability, investment, net issuance, beta, variance, accruals, short-term reversal, long-term reversal, and momentum
- Initial model always includes the market

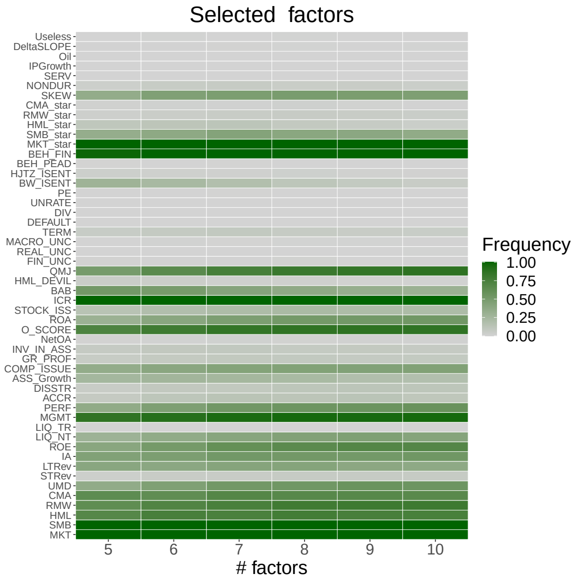
Robustness check: Model identification frequency



Post-screening factor space dimension



Factor selection frequency



Tradable factor risk premia